

Real numerical shadow and generalized B-splines

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Abstract

Restricted numerical shadow $P_A^X(z)$ of an operator A of order N is a probability distribution supported on the numerical range $W_X(A)$ restricted to a certain subset X of the set of all pure states – normalized, one-dimensional vectors in \mathbb{C}^N . Its value at point $z \in \mathbb{C}$ equals to the probability that the inner product $\langle u|A|u \rangle$ is equal to z , where u stands for a random complex vector from the set X distributed according to the natural measure on this set, induced by the unitarily invariant Fubini–Study measure. For a Hermitian operator A of order N we derive an explicit formula for its shadow restricted to real states, $P_A^{\mathbb{R}}(x)$, show relation of this density to the Dirichlet distribution and demonstrate that it forms a generalization of the B -spline. Furthermore, for operators acting on a space with tensor product structure, $\mathcal{H}_A \otimes \mathcal{H}_B$, we analyze the shadow restricted to the set of maximally entangled states and derive distributions for operators of order $N = 4$.

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1. Introduction

Consider a complex square matrix A of order N . Its standard *numerical range* is defined as the following subset of the complex plane,

$$W(A) = \{ \langle u|A|u \rangle : u \in \mathbb{C}^N, \|u\| = 1 \},$$

where u denotes a normalized complex vector in \mathcal{H}_N . Due to the Toeplitz–Hausdorff theorem this set is convex, while for a Hermitian A it forms an interval belonging to the real axis – see e.g. [1, 2, 3].

Among numerous generalizations of this notion we will be concerned with the *restricted numerical range*,

$$W_X(A) = \{\langle u|A|u \rangle : u \in \omega_X\}, \quad (1)$$

where ω_X forms a certain subset of the set ω of normalized complex vectors of size N . For instance, one can choose ω_X as the set of all real vectors, and analyze the 'real shadow' of A , denoted by $W_{\mathbb{R}}(A)$. For an operator A acting on a composed space, one studies also numerical range restricted to tensor product states, $W_{\otimes}(A)$, and the range $W_E(A)$ restricted to maximally entangled states [4, 5]. It is worth to emphasize a crucial difference with respect to the standard notion: the restricted numerical range needs not to be convex.

In order to define a probability measure supported on numerical range of $W(A)$ it is sufficient to consider the uniform measure on the sphere S^{2N-1} and the measure induced by the map $u \rightarrow \langle u|A|u \rangle \in W(A)$ [6, 7]. Alternatively, one considers the space of quantum states – equivalence classes of normalized vectors in \mathbb{C}^N , which differ by a complex phase, $u \sim e^{i\alpha}u$, and works with the Haar measure invariant under the action of the unitary group [8]. For any matrix A one defines in this way a probability measure $P_A(z)$ supported on $W(A)$ and called *numerical shadow* [6] or *numerical measure* [7]. The former name is inspired by the fact that for a normal matrix this measure can be interpreted as a shadow of an uniformly covered $(N-1)$ dimensional regular simplex projected on a plane [8, 9]. In a similar fashion, one can consider numerical shadow of matrices over the quaternion field, defined as the pushforward measure of the uniform measure on the sphere S^{4N-1} .

Even though several papers on numerical shadow were published during the last five years [6, 7, 8], the idea to associate with the numerical range a probability measure is much older: as described in a recent review by Holbrook [10] it goes back to the early papers of Davis [1].

Another variant of the numerical shadow of A can be obtained by taking random points from the subset ω_X of the set of pure states. The corresponding probability measure $P_A^X(z)$, called *restricted numerical shadow* [11], is by definition supported in restricted numerical range $W_X(A)$. More generally, one may take an arbitrary probability measure μ on the set of all pure states (or on the hypersphere S^{2N-1}) and study the measure induced in the numerical range of A .

Let A denote a Hermitian matrix of size N , so its numerical range is an interval on the real axis. The probability distribution generated by the map $\eta \mapsto \langle \eta|A|\eta \rangle$, where η is a random point on the unit sphere $\left\{ \eta \in \mathbb{C}^N : \sum_{j=1}^N |\eta_j|^2 = 1 \right\}$ equipped with the unitary-invariant surface measure, is then equal to the *shadow* of A .

It will be convenient to introduce the set Ω_N containing density matrices of order N , i.e. Hermitian positive definite operators, normalized by the trace condition, $\rho^* = \rho \geq 0$ with $\text{Tr} \rho = 1$. The set Ω_N is convex as it can be considered as the convex hull of the set of projectors on the pure states of dimension N – see e.g. [12]. Specifying a measure μ on the set of density matrices allows us to

propose a more general definition of numerical shadow.

Definition 1. For a given $N \times N$ matrix A and a probability measure μ on the space Ω_N of density matrices of order N we define the numerical shadow of matrix A with respect to μ as function on complex numbers

$$\mathcal{P}_A^\mu(z) = \int_{\Omega_N} d\mu(\rho) \delta(z - \text{Tr} A \rho). \quad (2)$$

The standard numerical shadow, defined in [8] and denoted by $\mathcal{P}_A(z)$, fulfills the above definition with μ supported on a pure states invariant to unitary transformations. In fact all restricted numerical shadow presented in [11] can be written in the above form.

The main goal of this work is to describe restricted numerical shadow for several relevant cases. For any symmetric real matrix A we derive its real numerical shadow. To this end we use Dirichlet distributions, the properties of which are reviewed in Sec. 2. We demonstrate that in this case the real shadow has the same distribution as a linear combination of components of a random vector generated by the Dirichlet distribution.

In Sec. 3 we briefly discuss B -splines, which correspond to complex shadows of Hermitian matrices, and show their link to generalized Dirichlet distributions. Complex and real shadows of illustrative normal matrices are compared in Sec. 4, in which some results are obtained for the case of Hermitian matrices.

Main result of this work — Theorem 14, which characterizes the real shadow of real symmetric matrices, is presented in Sec. 5. Continuity of the shadow at knots is discussed in Sec. 6, while formulae for the shadow with respect of real maximally entangled states for any matrix of size $N = 4$ are derived in Sec. 7.

2. The Dirichlet Distribution

Let \mathbb{T}_{N-1} in \mathbb{R}^{N-1} denotes the unit simplex of N -point probability distributions,

$$\mathbb{T}_{N-1} := \left\{ (t_1, \dots, t_{N-1}) \in \mathbb{R}^{N-1} : t_i \geq 0 \forall i, \sum_{i=1}^{N-1} t_i \leq 1 \right\}. \quad (3)$$

The *Dirichlet* distribution is a measure $\mu_{\mathbf{k}}$ on the simplex \mathbb{T}_{N-1} parameterized by a vector \mathbf{k} of N real numbers $k_1, \dots, k_N > 0$,

$$d\mu_{\mathbf{k}} = \frac{\Gamma\left(\sum_{i=1}^N k_i\right)}{\prod_{i=1}^N \Gamma(k_i)} \prod_{i=1}^{N-1} t_i^{k_i-1} \left(1 - \sum_{i=1}^{N-1} t_i\right)^{k_N-1} dt_1 \dots dt_{N-1}. \quad (4)$$

Note that the choice $\mathbf{k} = \{1, 1, \dots, 1\}$ gives the flat, Lebesgue measure on the simplex, while the case $\mathbf{k} = \{1/2, 1/2, \dots, 1/2\}$ corresponds to the statistical distribution – see e.g. [12].

Set $\tilde{k} := \sum_{i=1}^N k_i$. For $\alpha \in \mathbb{N}_0^N$ let $\alpha! := \prod_{i=1}^N \alpha_i!$, $|\alpha| := \sum_{i=1}^N \alpha_i$ and $t^\alpha := \prod_{i=1}^{N-1} t_i^{\alpha_i} \left(1 - \sum_{i=1}^{N-1} t_i\right)^{\alpha_N}$. It follows from the Dirichlet integral that

$$\int_{\mathbb{T}_{N-1}} t^\alpha d\mu_{\mathbf{k}} = \frac{1}{\binom{\tilde{k}}{|\alpha|}} \prod_{i=1}^N (k_i)_{\alpha_i}, \quad (5)$$

where $(x)_n := \prod_{i=1}^n (x + i - 1)$ denotes the Pochhammer product. It satisfies an important asymptotic relationship: $(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)} \sim z^n$ as $x \rightarrow \infty$ in the complex half-plane $\{z : \operatorname{Re} z > 0\}$.

Consider the random vector corresponding to choosing a point in \mathbb{T}_{N-1} according to $d\mu_{\mathbf{k}}$ with components (T_1, \dots, T_N) with $T_N := 1 - \sum_{i=1}^{N-1} T_i$. We select an arbitrary vector of N real numbers ordered increasingly, $a_1 \leq a_2 \leq \dots \leq a_N$, and will be concerned with the probability distribution of their weighted average,

$$X = \sum_{i=1}^N a_i T_i.$$

Definition 2. *The distribution of random variable X will be denoted as*

$$\mathcal{D}(a_1, \dots, a_N; k_1, \dots, k_N). \quad (6)$$

In the case some values of a_i are repeated some formulae have to be modified. It is clear that $a_1 \leq X \leq a_N$. There is a moment generating function for X .

Let $F(x)$ denote the cumulative distribution function of X , that is,

$$F(x) := \Pr\{X \leq x\}. \quad (7)$$

Lemma 3. *Suppose $|r| < \min_i \frac{1}{|a_i|}$ then*

$$\mathcal{E}\left[(1 - rX)^{-\tilde{k}}\right] = \prod_{i=1}^N (1 - ra_i)^{-k_i}. \quad (8)$$

This Lemma, proof of which is provided in Appendix B, implicitly gives an expression for the moments, $\mathcal{E}[X^n]$, because $(1 - rX)^{-\tilde{k}} = \sum_{n=0}^{\infty} \frac{\binom{\tilde{k}}{n}}{n!} r^n X^n$.

Corollary 4. *The mean $\mu := \mathcal{E}[X] = \frac{1}{\tilde{k}} \sum_{i=1}^N k_i a_i$ and the variance $\mathcal{E}\left[(X - \mu)^2\right] = \frac{1}{\tilde{k}(\tilde{k}+1)} \sum_{i=1}^N k_i (a_i - \mu)^2$.*

In the case of $N = 2$ it is straightforward to find the density for $\mathcal{D}(a_1, a_2; k_1, k_2)$,

$$f(x) = \frac{1}{B(k_1, k_2)(a_2 - a_1)^{k_1+k_2-1}} (a_2 - x)^{k_1-1} (x - a_1)^{k_2-1}, \quad (9)$$

where $B(a, b)$ denotes the beta function.

Let us now return to the general case of an arbitrary dimension N and consider the behavior of $F(x)$ for $x \notin \{a_1, a_2, \dots, a_N\}$. Here we require no repeated values in $\{a_i\}$. This involves the intersection of the hyperplane $\sum_{i=1}^N a_i t_i = x$ with \mathbb{T}_{N-1} , which is a convex polytope whose faces are subsets of $\pi_i := \{t : t_i = 0\}$ for $1 \leq i \leq N-1$, $\pi_N := \{t : \sum_{i=1}^{N-1} t_i = 1\}$, and $\pi_x := \{t : \sum_{i=1}^N a_i t_i = x\}$. Note that $\sum_{i=1}^N a_i t_i = x$ is equivalent to $\sum_{i=1}^{N-1} (a_N - a_i) t_i = a_N - x$. The vertices of this polytope come from the intersection of $N-2$ hyperplanes drawn from $\{\pi_i : 1 \leq i \leq N\}$ with π_x . Introduce the unit basis vectors ε_i ($1 \leq i \leq N-1$) with components (δ_{ij}) . There are two types of vertices:

$$\xi_i(x) = \bigcap_{j=1, j \neq i}^{N-1} \pi_j \cap \pi_x = \frac{a_N - x}{a_N - a_i} \varepsilon_i, 1 \leq i \leq N-1; \quad (10)$$

$$\xi_{ij}(x) = \bigcap_{\ell=1, \ell \neq i, j}^{N-1} \pi_\ell \cap \pi_N \cap \pi_x = \frac{a_j - x}{a_j - a_i} \varepsilon_i + \frac{x - a_i}{a_j - a_i} \varepsilon_j, 1 \leq i < j \leq N-1. \quad (11)$$

For any given x some of these vertices are in \mathbb{T}_{N-1} and some are not. Suppose $a_M < x < a_{M+1}$ for some M with $1 \leq M < N$, then $\xi_i(x) \in \mathbb{T}_{N-1}$ exactly when $1 \leq i \leq M$ since the condition is $0 < \frac{a_N - x}{a_N - a_i} < 1$, that is, $x > a_i$. Similarly $\xi_{ij}(x) \in \mathbb{T}_{N-1}$ exactly when $a_i < x < a_j$, that is, $1 \leq i \leq M$ and $M+1 \leq j \leq N-1$. Thus the number of vertices is $M(N-M)$. Each vertex is an extreme point: to show this one exhibits a linear function $c_0 + \sum_{i=1}^{N-1} c_i t_i$ which vanishes at the point and is positive at all other vertices. For $\xi_i(x)$ the function $\sum_{j \neq i} t_j$ accomplishes this, and for $\xi_{ij}(x)$ use $1 - t_i - t_j$ (this applies to the vertices contained in \mathbb{T}_{N-1} , by inspection).

Remark 5. Suppose $a_M < x_1 < x_2 < a_{M+1}$ then $F(x_2) - F(x_1)$ is given by the integral of $d\mu_{\mathbf{k}}$ over a convex polytope with $2M(N-M)$ vertices lying between parallel hyperplanes. The vertices of the polytope are analytic functions of x and so $F(x_2) - F(x_1)$ is analytic in x_2 and in the parameters k_1, k_2, \dots, k_N (in broad terms, decompose the integral as a sum of iterated $(N-1)$ -fold integrals each of which has an analytic expression).

It is straightforward to find the following infinite series expression for the complementary distribution function $1 - F(x)$ for $x \in (a_{N-1}, a_N]$ – see Appendix B. We assumed here that $a_{N-1} < a_N$, but other repetitions are allowed.

Proposition 6. For $a_{N-1} < x \leq a_N$

$$\begin{aligned} 1 - F(x) &= \frac{\Gamma(\tilde{k}) (a_N - x)^{\tilde{k} - k_N}}{\Gamma(k_N) \Gamma(\tilde{k} - k_N)} \prod_{i=1}^{N-1} (a_N - a_i)^{-k_i} \\ &\times \sum_{\alpha \in \mathbb{N}_0^{N-1}} \frac{(1 - k_N)_{|\alpha|}}{(\tilde{k} - k_N)_{|\alpha|+1}} (a_N - x)^{|\alpha|} \prod_{i=1}^{N-1} \frac{(k_i)_{\alpha_i}}{\alpha_i! (a_N - a_i)^{\alpha_i}}. \end{aligned}$$

Corollary 7. For x near a_N (and $x < a_N$) $1 - F(x)$ behaves like $(a_N - x)^{\tilde{k} - k_N}$ and the density $f(x) = \frac{d}{dx} F(x)$ behaves like $(a_N - x)^{\tilde{k} - k_N - 1}$.

The Dirichlet distribution has a special additivity property which allows us to restrict to the situation where the a_i 's are mutually distinct. If two numbers a_i 's are equal, say $a_{N-1} = a_N$ then $\sum_{i=1}^N a_i t_i$ has the same distribution as $\mathcal{D}(a_1, \dots, a_{N-1}; k_1, \dots, k_{N-1} + k_N)$ (see 6). In other words if $a_\ell = a_{\ell+1} = \dots = a_{\ell+m-1}$ then the distribution is the same as

$$\mathcal{D}\left(a_1, \dots, a_\ell, a_{\ell+m}, \dots, a_N; k_1, \dots, \sum_{i=\ell}^{\ell+m-1} k_i, k_{\ell+m}, \dots, k_N\right). \quad (12)$$

When each k_i is an integer (≥ 1) there is a finite sum expression for the density in terms of piecewise polynomials (splines). This theorem is from [6, p.2070]. For simplicity we state the result for the case $0 \leq a_1 < a_2 < \dots < a_N$. Let $x_+ := \max(0, x)$, with the convention that $x_+^0 = 1$ for $x \geq 0$ and $= 0$ for $x < 0$.

Theorem 8. Suppose $0 \leq a_1 < a_2 < \dots < a_N$, $k_i \in \mathbb{N}$ for each i , then

$$f(x) = \sum_{i=1}^N \sum_{j=1}^{k_i} \frac{\beta_{ij}}{a_i B(j, \tilde{k} - j)} \left(\frac{x}{a_i}\right)_+^{j-1} \left(1 - \frac{x}{a_i}\right)_+^{\tilde{k} - j - 1}, \quad (13)$$

where

$$\prod_{i=1}^N (1 - ra_i)^{-k_i} = \sum_{i=1}^N \sum_{j=1}^{k_i} \frac{\beta_{ij}}{(1 - ra_i)^j} \quad (14)$$

is the partial fraction decomposition (the term with $i = 1$ is omitted if $a_1 = 0$).

Observe that each term $\frac{1}{a_i B(j, \tilde{k} - j)} \left(\frac{x}{a_i}\right)_+^{j-1} \left(1 - \frac{x}{a_i}\right)_+^{\tilde{k} - j - 1}$ is itself a probability density supported on $0 \leq x \leq a_i$. (In the present context N is the number of distinct values, differing from the statement in [6] where each $k_i = 1$ and some values are repeated.) The Theorem shows that the density is a piecewise polynomial of degree $\tilde{k} - 2$ with discontinuities (in some order derivative) at the points $\{a_i\}$. Because of this spline interpretation the quantities a_i will henceforth be called *knots*.

3. B-splines and their generalization

The Dirichlet distribution is closely related to the notion of an s -dimensional B -spline introduced by de Boor [13].

Definition 9. Let σ be a non-trivial simplex in \mathbb{R}^{s+k} . On \mathbb{R}^s we define the B -spline of order k from σ by

$$\mathcal{M}_{k,\sigma}(x_1, \dots, x_s) = \text{vol}(\sigma \cap \{v \in \mathbb{R}^{s+k} : v_j = x_j \ (j = 1, 2, \dots, s)\}). \quad (15)$$

A measure version of the above definition is more useful, thus we define the normalized measure on \mathbb{R}^s

$$\mathcal{M}_{k,\sigma}(B) = \text{vol}(\sigma \cap \{v \in \mathbb{R}^{s+k} : \{v_j\}_{j=1}^s \in B\}) / \text{vol}(\sigma). \quad (16)$$

A non-trivial simplex $\sigma \in \mathbb{R}^{s+k}$ can be written as $W\mathbb{T}_{s+k}$ where \mathbb{T}_{s+k} is a regular simplex and W is an invertible matrix of order $s+k$. The simplex is possibly translated if 0 is not a vertex of σ . We will use the notation

$$\mathcal{M}_{k,W}(B) = \text{vol}(y \in \mathbb{T}_{s+k} : Wy \in B \oplus \mathbb{R}^k) / \text{vol}(\mathbb{T}_{s+k}). \quad (17)$$

Instead of calculating the volume with respect to the flat Lebesgue measure one can use instead the Dirichlet measure $\mu_{\mathbf{k}}$ with parameters \mathbf{k} instead. In this way one obtains a generalized notion of B -splines.

$$\mathcal{M}_{k,W}^{(\mathbf{k})}(B) = \mu_{(\mathbf{k})}(y \in \mathbb{T}_{s+k} : Wy \in B \oplus \mathbb{R}^k) / \mu_{(\mathbf{k})}(\mathbb{T}_{s+k}). \quad (18)$$

Therefore, the distribution \mathcal{D} can be viewed as a generalized B -spline. If we take any $N \times N$ invertible matrix W with the first row given by $\lambda_1 \dots \lambda_N$, then a generalized B -spline is equal to the distribution \mathcal{D}

$$\mathcal{M}_{N-1,W}^{(\mathbf{k})} = \mathcal{D}(\lambda_1, \dots, \lambda_N; \mathbf{k}). \quad (19)$$

4. Shadows of Hermitian and real symmetric matrices

Among several probability measures defined on the set of density matrices it is convenient to distinguish a class of measures induced by the partial trace performed on a pure state on the extended system.

We say, that a density matrix ρ of size N is distributed according to the induced measure $\mu_{N,K}^{\text{tr}}$ [12] if

$$\rho = \text{Tr}_2 |\psi\rangle\langle\psi|, \quad (20)$$

where $|\psi\rangle$ being a uniformly distributed, normalized random vector in $\mathcal{H}_1 \otimes \mathcal{H}_2 = \mathbb{C}^N \otimes \mathbb{C}^K$ and the operation of partial trace is defined for product matrices as $\text{Tr}_2 A \otimes B = A \text{Tr} B$ and extended to general case by linearity. In the case of $K = 1$ we obtain a measure on pure states and in the case of $K = N$ we get a Hilbert-Schmidt measure [12].

In paper [6] we showed that the (complex) shadow of a Hermitian matrix A with eigenvalues $(\lambda_1, \dots, \lambda_N)$ (counted with multiplicity) has the distribution

$$\mathcal{P}_A = \mathcal{D}(\lambda_1, \dots, \lambda_N; 1, \dots, 1). \quad (21)$$

From Corollary 4 the mean is $\mu = \frac{1}{N} \sum_{j=1}^N \lambda_j = \frac{1}{N} \text{Tr} A$ and the variance is $\frac{1}{N(N+1)} \sum_{j=1}^N (\lambda_j - \mu)^2$.

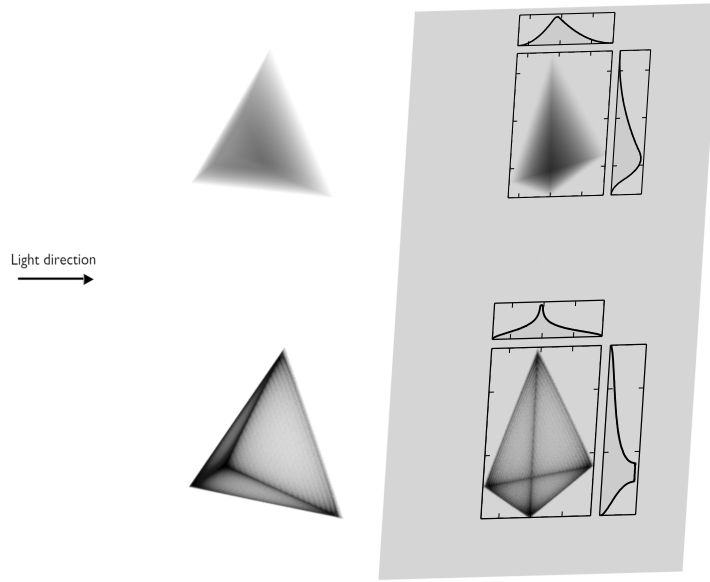


Figure 1: Numerical shadows of an illustrative normal matrix A of order $N = 4$. Upper figure represents the complex shadow, while the lower one the real shadow. The tetrahedrons on the left are covered with respect to the uniform and the Dirichlet distribution, respectively. If a parallel beam of light is shined upon them they cast shadows on a plane which coincide with complex and real numerical shadows of A . Both marginal distributions above and on the right hand side of each shadow, correspond to complex/real numerical shadows of Hermitian matrices formed from real and imaginary parts of A .

In analogy to the standard shadow (21) one can introduce the *mixed states shadow* [8]. For a Hermitian matrix A the mixed shadow induced by a distribution $\mu_{N,K}^{\text{tr}}$ reads

$$\mathcal{P}_A^K = \mathcal{D}(\lambda_1, \dots, \lambda_N; K, \dots, K). \quad (22)$$

This follows directly from the definition of a partial trace and the additivity property of a Dirichlet distribution. As a special case we obtain, that the mixed numerical shadow with respect to flat Hilbert Schmidt distribution is given by $\mathcal{P}_A^N = \mathcal{D}(\lambda_1, \dots, \lambda_N; N, \dots, N)$. We can calculate mean and variance for mixed numerical shadow induced by $\mu_{N,K}^{\text{tr}}$, using Corollary 4 we have $\mu = \frac{1}{N} \sum_{j=1}^N \lambda_j = \frac{1}{N} \text{Tr} A$ and the variance is $\frac{1}{N(NK+1)} \sum_{j=1}^N (\lambda_j - \mu)^2$.

Let us now return to the main subject of the paper - the shadow $\mathcal{P}_A^{\mathbb{R}}$ of a matrix A of order N with respect to the set of real pure states in \mathbb{R}^N . It is briefly called the *real shadow* [11], and for a real symmetric matrix A it can be related to the Dirichlet distribution,

$$\mathcal{P}_A^{\mathbb{R}} = \mathcal{D}\left(\lambda_1, \dots, \lambda_N; \frac{1}{2}, \dots, \frac{1}{2}\right) \quad (23)$$

where $\lambda_1, \dots, \lambda_N$ denotes the eigenvalues of A counted with multiplicity. The mean value is $\mu = \frac{1}{N} \sum_{j=1}^N \lambda_j$ and the variance is $\frac{2}{N(N+2)} \sum_{j=1}^N (\lambda_j - \mu)^2$.

In a close analogy to the complex case, one can also consider the shadow with respect to real mixed states obtained by an induced measure $\mu_{N,K}^{\text{tr}}$. For any real symmetric matrix A this leads to the distribution \mathcal{D} , with all indices equal to $K/2$. Thus the real shadow is obtained for $K = 1$, as required.

Henceforth we will concentrate on the distributions $\mathcal{D}(a_1, \dots, a_N; k, \dots, k)$ with pairwise distinct knots a_i . For integer k we have the interpretation as the shadow of the $Nk \times Nk$ Hermitian matrix $A \oplus \dots \oplus A$ (k summands) where the eigenvalues of A are a_1, \dots, a_N , or the mixed numerical shadow induced by the measure $\mu_{N,k}^{\text{tr}}$. We consider the distribution as an analytic function of k , for $\text{Re } k > 0$, and will find more information by extrapolating from the known formulas for integer k . Start with finding explicit values of the coefficients $\{\beta_{ij}\}$ in Theorem 8.

Lemma 10. *Suppose $\{a_1, a_2, \dots, a_N\}$ consists of pairwise distinct nonzero real numbers and $k = 1, 2, 3, \dots$ then*

$$\prod_{i=1}^N (1 - ra_i)^{-k} = \sum_{i=1}^N \sum_{m=0}^{k-1} \frac{(-1)^m a_i^{(N-1)k}}{(1 - ra_i)^{k-m}} \sum_{\alpha \in \mathbb{N}_0^N, |\alpha|=m, \alpha_i=0} \frac{1}{\alpha!} \prod_{j=1, j \neq i}^N \frac{(k)_{\alpha_j} a_j^{\alpha_j}}{(a_i - a_j)^{k+\alpha_j}} \quad (24)$$

The proof is provided in Appendix B.

Thus the formula in Theorem 8 is completely symmetric in (a_1, a_2, \dots, a_N) , independent of the ordering. This is an ingredient in the derivation of the differential equation satisfied by the density.

Consider the case of a symmetric matrix of size $N = 3$. Then the density for its real shadow has an expression in terms of a ${}_2F_1$ -hypergeometric function which solves a certain second-order differential equation. Suppose $a_1 = 0$, so formulas (14) and (10) (change j to $k - m$) read for $x \in (a_2, x \leq a_3]$

$$f(x) = \frac{x^{k-1} (a_3 - x)^{2k-1}}{B(k, 2k) a_3^{2k-1} (a_3 - a_2)^k} {}_2F_1 \left(\begin{matrix} k, 1-k \\ 2k \end{matrix}; \frac{a_2 (a_3 - x)}{x (a_3 - a_2)} \right); \quad (25)$$

and the series converges for any $k > 0$.

Let us now return to the generalized case of an arbitrary matrix order N , for which condition $a_1 < a_2 < \dots < a_N$ holds. Basing on computational experiments we are in position to formulate a generalization valid for small N and integers k . Set $P_N(x) = \prod_{i=1}^N (x - a_i)$ and define a differential operator \mathcal{T}_k of order $N - 1$ (with $\partial := \frac{d}{dx}$) by

$$\mathcal{T}_k := P_N(x) \partial^{N-1} + \sum_{j=1}^{N-1} (-1)^j \frac{N-j}{N} \frac{(N(k-1))_j}{j!} \partial^j P_N(x) \partial^{N-1-j}. \quad (26)$$

The differential equation $\mathcal{T}_k f(x) = 0$ has regular singular points at the knots. We will show that the density function of $\mathcal{D}(a_1, \dots, a_N; k, \dots, k)$ satisfies this equation at all $x \notin \{a_1, \dots, a_N\}$, first for integer k then for $k > 0$. The idea is to verify the equation for the interval (a_{N-1}, a_N) by use of Proposition 6 and then use the symmetry property of Theorem 8 to extend the result to all intervals (a_i, a_{i+1}) .

Lemma 11. *For arbitrary a, b, c and $n = 1, 2, \dots$*

$$\sum_{j=0}^n \frac{a+j}{a} \frac{(-n)_j}{(c)_j} \frac{(b)_j}{j!} = \frac{(c-b)_{n-1}}{a(c)_n} (a(c-b+n-1) - nb). \quad (27)$$

Proof. Expand the sum as

$$\begin{aligned} \sum_{j=0}^n \frac{(-n)_j}{(c)_j} \frac{(b)_j}{j!} + \frac{1}{a} \sum_{j=1}^n \frac{j(-n)_j}{(c)_j} \frac{(b)_j}{j!} &= \sum_{j=0}^n \frac{(-n)_j}{(c)_j} \frac{(b)_j}{j!} - \frac{nb}{ac} \sum_{i=0}^{n-1} \frac{(1-n)_i}{(c+1)_i} \frac{(b+1)_i}{i!} \\ &= \frac{(c-b)_n}{(c)_n} - \frac{nb(c-b)_{n-1}}{ac(c+1)_{n-1}}, \end{aligned} \quad (28)$$

by the Chu-Vandermonde sum. ■

Since we intend to work with polynomials in $x - a_N$ we set $y := x - a_N$. Start the verification by replacing $P_N(x)$ by y^n and apply the resulting operator to $(-y)^c$ (for $0 \leq n \leq N - 1$ and generic c (leaving open the possibility of c being a noninteger and $y < 0$). At times we use the Pochhammer symbol with a negative index: for $m = 1, 2, 3, \dots$ let $(c)_{-m} = 1/(c-m)_m$, so that

$(c)_{-m}(c-m)_m = (c)_0 = 1$. Note that $\partial^j (-y)^c = (-c)_j (-y)^{c-j}$, so the result follows

$$\begin{aligned}
& \sum_{j=0}^{N-1} (-1)^j \frac{N-j}{N} \frac{(N(k-1))_j}{j!} (-1)^j (-n)_j y^{n-j} (-c)_{N-1-j} (-y)^{c-N+1+j} \\
&= (-1)^n (-y)^{c+n-N+1} (-c)_{N-1} \sum_{j=0}^n \frac{j-N}{-N} \frac{(-n)_j (N(k-1))_j}{j! (c+2-N)_j} \\
&= (-1)^n (-y)^{c+n-N+1} \frac{(-c)_{N-1}}{(c+2-N)_n} (c+2-Nk)_{n-1} (c+1-(N-n)k) \\
&= (-y)^{c+n-N+1} (-c)_{N-1-n} (c+2-Nk)_{n-1} (c+1-(N-n)k).
\end{aligned} \tag{29}$$

In the special case $n = N$ we obtain $-(-y)^{c+1} (c+2-Nk)_{N-1}$ since $(-c)_{-1} = -1/(1+c)$. For $n = 0$ the result is zero. The calculations used the reversal $(a)_{m-j} = (-1)^j \frac{(a)_m}{(1-m-a)_j}$ and the Lemma with a, b, c replaced by $-N, Nk - N$ and $c+2-N$, respectively. The upper limit of summation is n because $n \leq N$. To proceed further we introduce:

$$A(N, k, n, c) := (-c)_{N-1-n} (c+2-Nk)_{n-1} (c+1-(N-n)k), 1 \leq n < N; \tag{30}$$

$$A(N, k, N, c) := -(c+2-Nk)_{N-1}.$$

Next $P_N(x) = y \prod_{i=1}^{N-1} (y - (a_i - a_N)) = \sum_{j=0}^{N-1} (-1)^{N-1-j} e_{N-1-j} y^{j+1}$ where e_m denotes the elementary symmetric polynomial of degree m in $\{a_1 - a_N, \dots, a_{N-1} - a_N\}$, $0 \leq m \leq N-1$. Thus

$$\mathcal{T}_k((-y)^c) = \sum_{j=1}^N (-1)^{N-j} e_{N-j} A(N, k, j, c) (-y)^{c-N+j+1}. \tag{31}$$

Up to a multiplicative constant, not relevant in this homogeneous equation, the density in $a_{N-1} < x < a_N$ is given by

$$f_0(x) = -\partial \sum_{\alpha \in \mathbb{N}_0^{N-1}} \frac{(1-k)_{|\alpha|}}{((N-1)k)_{|\alpha|+1}} (-y)^{|\alpha|+(N-1)k} \prod_{i=1}^{N-1} \frac{(k)_{\alpha_i}}{\alpha_i! (a_N - a_i)^{\alpha_i}}. \tag{32}$$

The series terminates at $|\alpha| = k-1$. Define symmetric polynomials $S_m(k; a)$ in $\{a_1 - a_N, \dots, a_{N-1} - a_N\}$ (note the reversal to $a_i - a_N$) by

$$\sum_{m=0}^{\infty} S_m(k; a) r^m = \prod_{i=1}^{N-1} \left(1 - \frac{r}{a_i - a_N}\right)^{-k}, \tag{33}$$

convergent for $|r| < a_N - a_{N-1}$, then

$$f_0(x) = \sum_{m=0}^{k-1} \frac{(1-k)_m}{((N-1)k)_m} (-y)^{(N-1)k+m-1} (-1)^m S_m(k; a), \tag{34}$$

and

$$\begin{aligned}
\mathcal{T}_k f_0(x) &= \sum_{m=0}^{k-1} \sum_{j=1}^N (-1)^{N-j} e_{N-j} \frac{(1-k)_m}{((N-1)k)_m} S_m(k; a) (-1)^m \\
&\quad \times A(N, k, j, (N-1)k + m - 1) (-y)^{(N-1)k + m - N + j} \\
&= \sum_{\ell=1}^{N+k-1} (-y)^{(N-1)k - N + \ell} \sum_{j=1}^{\min(N, \ell)} (-1)^{N-j} e_{N-j} \frac{(1-k)_{\ell-j}}{((N-1)k)_{\ell-j}} \\
&\quad \times (-1)^{\ell-j} S_{\ell-j}(k; a) A(N, k, j, (N-1)k + \ell - j - 1).
\end{aligned} \tag{35}$$

It is required to show that the j -sum vanishes for each ℓ . At $\ell = 1, j = 1$ there is only one term and $A(N, k, 1, (N-1)k - 1) = 0$. Replace $A(\cdot)$ by its definition (30) and simplify

$$\begin{aligned}
&\frac{(1-k)_{\ell-j}}{((N-1)k)_{\ell-j}} (1+j-\ell-(N-1)k)_{N-1-j} (1-k+\ell-j)_{j-1} (\ell-j+k(j-1)) \\
&= (-1)^{\ell-j} (1-k)_{\ell-1} (\ell-j+k(j-1)) \frac{(1+j-\ell-(N-1)k)_{N-1-j}}{(1+j-\ell-(N-1)k)_{\ell-j}} \\
&= (-1)^{\ell-j} (1-k)_{\ell-1} (\ell-j+k(j-1)) \frac{(1-\ell-(N-1)k)_{N-1}}{(1-\ell-(N-1)k)_{\ell}}.
\end{aligned} \tag{36}$$

The denominator does not vanish because $(N-1)k > 0$. Taking out the factors depending only on ℓ the j -sum becomes

$$\sum_{j=1}^{\min(N, \ell)} (-1)^j (\ell-j+k(j-1)) e_{N-j} S_{\ell-j}(k; a). \tag{37}$$

There is a recurrence relation for $S_m(k; a)$; the elementary symmetric function of degree m in $\left\{ \frac{1}{a_1 - a_N}, \dots, \frac{1}{a_{N-1} - a_N} \right\}$ equals $\frac{e_{N-1-m}}{e_{N-1}}$ for $0 \leq m \leq N-1$. The generating function of $\{S_m(k; a)\}$ is $g(r)^{-k}$ where

$$g(r) := \prod_{j=1}^{N-1} \left(1 - \frac{r}{a_j - a_N} \right) = \sum_{i=0}^{N-1} (-1)^i \frac{e_{N-1-i}}{e_{N-1}} r^i. \tag{38}$$

Extract the coefficient of r^m in the following equation

$$\begin{aligned}
g(r) \frac{\partial}{\partial r} [g(r)^{-k}] &= -k \left(\frac{\partial}{\partial r} g(r) \right) g(r)^{-k} \\
\sum_{i=0}^{N-1} (-1)^i \frac{e_{N-1-i}}{e_{N-1}} r^i \sum_{j=0}^{\infty} j S_j(k; a) r^{j-1} &= -k \sum_{i=0}^{N-1} i (-1)^i \frac{e_{N-1-i}}{e_{N-1}} r^{i-1} \sum_{j=0}^{\infty} S_j(k; a) r^j,
\end{aligned} \tag{39}$$

to obtain

$$\sum_{i=0}^{\min(m+1, N-1)} (-1)^i (m-i+1+ki) \frac{e_{N-1-i}}{e_{N-1}} S_{m-i+1}(k; a) = 0. \tag{40}$$

Now set $i = j - 1$ and $m = \ell - 2$ (recall the case $\ell = 1$ was already done) to show that the expression in (37) vanishes.

Theorem 12. *Suppose $k = 1, 2, 3, \dots$ then the density $f(x)$ of $\mathcal{D}(a_1, \dots, a_N; k, \dots, k)$ satisfies the linear differential equation $\mathcal{T}_k f(x) = 0$ at all $x \notin \{a_1, \dots, a_N\}$.*

Proof. Assume first that $a_1 > 0$. The above argument showed that $\mathcal{T}_k f(x) = 0$ for $a_{N-1} < x < a_N$. On this interval $f(x)$ is a constant multiple of

$$p_N(x) := \sum_{j=1}^k \beta_{Nj} \frac{1}{B(Nk-j, j) a_N} \left(\frac{x}{a_N}\right)_+^{j-1} \left(1 - \frac{x}{a_N}\right)_+^{Nk-j-1}, \quad (41)$$

– see Theorem 8. Because $\mathcal{T}_k p_N(x) = 0$ is a polynomial equation it holds for all $x \neq 0, a_N$. The piecewise polynomial p_N has coefficients which are symmetric in a_1, \dots, a_{N-1} – see equation (10). Hence the differential equation is symmetric in (a_1, \dots, a_N) and each piece

$$p_i(x) := \sum_{j=1}^k \beta_{ij} \frac{1}{B(Nk-j, j) a_i} \left(\frac{x}{a_i}\right)_+^{j-1} \left(1 - \frac{x}{a_i}\right)_+^{Nk-j-1} \quad (42)$$

satisfies the differential equation for $x \neq 0, a_i$. The density is the sum $\sum_{i=1}^N p_i$ thus $\mathcal{T}_k f(x) = 0$ at each $x \notin \{a_1, \dots, a_N\}$. The density $f_c(x)$ of $\mathcal{D}(a_1 + c, \dots, a_N + c; k, \dots, k)$ equals the translate $f(x - c)$ and the differential operator \mathcal{T}_k has a corresponding translation property and thus the restriction $a_1 > 0$ can be removed. ■

Corollary 13. *If $k > 0$ then the density $f(x)$ of $\mathcal{D}(a_1, \dots, a_N; k, \dots, k)$ satisfies the linear differential equation $\mathcal{T}_k f(x) = 0$ at all $x \notin \{a_1, \dots, a_N\}$.*

Proof. Suppose $a_M < x_1 < x_2 < a_{M+1}$. The probability $\Pr\{x_1 < X < x_2\}$ is given by a definite integral with respect to an integrand which is analytic for $\operatorname{Re} k > 0$ over a polytope in \mathbb{T}_{N-1} whose vertices are independent of k and analytic in x_1, x_2 – see Remark 5. Thus the distribution function $F(x)$ at x is analytic for $\operatorname{Re} k > 0$ and extends to an analytic function in x for $x_1 < \operatorname{Re} x < x_2, |\operatorname{Im} x| < \varepsilon$ for some $\varepsilon > 0$. The differential equation $\mathcal{T}_k \frac{\partial}{\partial x} F(x) = 0$ is satisfied for each $k = 1, 2, 3, \dots$, this is an analytic relation and extends to all $\operatorname{Re} k > 0$ by Carlson's theorem (see Henrici [14, vol.2, p.334]). ■

We can now assert the validity of the equation for $k = \frac{1}{2}, \frac{3}{2}, \dots$ which applies to real shadows or the repeated eigenvalue case (each is repeated 3 times, or 5 times, etc.). It is not clear what happens if just one eigenvalue is repeated, note that the main result used symmetric functions of $\frac{1}{a_i - a_N}$. It is plausible that the equation applies in intervals adjacent to simple (non-repeated) eigenvalues.

The case $k = \frac{1}{2}$ is of special interest since it applies to the real shadow when the eigenvalues are pairwise distinct. The equation $\mathcal{T}_{1/2}f(x) = 0$ is

$$P_N(x) \partial^{N-1} f(x) + \sum_{j=1}^{N-1} (-1)^j \frac{N-j}{N} \frac{(-N/2)_j}{j!} \partial^j P_N(x) \partial^{N-1-j} f(x) = 0. \quad (43)$$

When N is even then the terms $\partial^m f(x)$ for $0 \leq m \leq \frac{N}{2} - 2$ drop out, that is any polynomial of degree $\frac{N}{2} - 2$ satisfies the equation. This property will be made precise in the next section.

The *indicial equation* is important because it provides information about the solutions in neighborhoods of the knots, that is, the solutions have the form

$$\sum_{n=0}^{\infty} \gamma_n (x - a_j)^{n+c}, \sum_{n=0}^{\infty} \gamma_n (a_j - x)^{n+c}, \quad (44)$$

(depending on whether the solution is valid for $x > a_j$ or $x < a_j$) where c is a solution of the indicial equation: this comes from the coefficient of the lowest power in $\mathcal{T}_k(a_N - x)^c$ from equation (31), namely

$$e_{N-1} A(N, k, 1, c) = (-1)^{N-1} e_{N-1} (-c)_{N-2} (c+1 - (N-1)k) = 0. \quad (45)$$

The solutions, called *critical exponents*, are $c = 0, 1, \dots, N-3, (N-1)k-1$. In the real shadow situation with $k = \frac{1}{2}$ we see there are two different types: when $N = 2m+1$ the critical exponent $c = m-1$ is repeated which leads to a logarithmic solution: $\sum_{n=0}^{\infty} \gamma_n (x - a_j)^{m-1+n}$ and $\log|x - a_j| \sum_{n=0}^{\infty} \gamma'_n (x - a_j)^{m-1+n}$. This actually occurs, as will be shown in the sequel.

5. The real shadow

We will use “heuristic extrapolation” to postulate a set of formulas for the real shadow (23) – the density of $\mathcal{D}(a_1, \dots, a_N; \frac{1}{2}, \dots, \frac{1}{2})$. In the notation of Theorem 12 there is a set of functions $p_j(x)$, with a symmetry property, such that the density $f(x) = \sum_{j=m}^N p_j(x)$ in the interval (a_{m-1}, a_m) . It is straightforward to do this in the top interval (a_{N-1}, a_N) but the expression involves square roots of quantities that become negative for $x < a_{N-1}$. The idea is to adopt certain branches of the complex square roots which give plausible results and then to prove the validity of the postulated formulas. This will be done by using complex contour integration to verify the known moment generating function.

We begin by pointing out that the expression for the density in (a_{N-1}, a_N) found in Proposition 6 is a multiple infinite series which diverges for $|x - a_N| > a_N - a_{N-1}$, not an easy expression to evaluate. We can replace it by a one-variable (definite) integral when $k = \frac{1}{2}$. Suppose the series $g(r) = \sum_{n=0}^{\infty} \gamma_n r^n$ converges for $|r| \leq 1$ then

$$\frac{1}{B(\frac{1}{2}, \frac{N}{2} - 1)} \int_0^1 \sum_{n=0}^{\infty} \gamma_n t^n t^{-1/2} (1-t)^{N/2-2} dt = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{(\frac{N-1}{2})_n} \gamma_n. \quad (46)$$

Apply this to

$$g(r) = \prod_{j=1}^{N-1} \left(1 - \frac{a_N - x}{a_N - a_j} r\right)^{-\frac{1}{2}} = \sum_{\alpha \in \mathbb{N}_0^{N-1}} \prod_{j=1}^{N-1} \frac{(k_i)_{\alpha_j}}{\alpha_j!} \left(\frac{a_N - x}{a_N - a_j}\right)^{a_j} r^{|\alpha|} \quad (47)$$

and use the formula for the density from Proposition 6 and act with $-\frac{\partial}{\partial x}$ on $1 - F(x)$ to obtain the density for $x \in (a_{N-1}, a_N)$,

$$f(x) = \frac{N-2}{2\pi(a_N - x)} \prod_{j=1}^{N-1} \left(\frac{a_N - x}{a_N - a_j}\right)^{\frac{1}{2}} \times \int_0^1 \prod_{j=1}^{N-1} \left(1 - \frac{a_N - x}{a_N - a_j} t\right)^{-\frac{1}{2}} t^{-1/2} (1-t)^{N/2-2} dt. \quad (48)$$

Note that $B\left(\frac{1}{2}, \frac{N}{2} - 1\right) B\left(\frac{1}{2}, \frac{N-1}{2}\right) = \frac{\Gamma(1/2)^2 \Gamma(N/2-1)}{\Gamma(N/2)} = \frac{\pi}{N/2-1}$. Make the change of variable $s = a_N - t(a_N - x)$, then

$$f(x) = \frac{N-2}{2\pi} \int_x^{a_N} (a_N - s)^{-\frac{1}{2}} \prod_{j=1}^{N-1} (s - a_j)^{-\frac{1}{2}} (s - x)^{\frac{N}{2}-2} ds. \quad (49)$$

Suppose we want to interpret this integral for $a_{N-2} < x < a_{N-1}$ then we must pick a branch of $(s - a_{N-1})^{-\frac{1}{2}}$, that is we need to choose the sign in $(s - a_{N-1})^{-\frac{1}{2}} = \pm i(a_{N-1} - s)^{-\frac{1}{2}}$, where $i = \sqrt{-1}$. Denote the integral by $f_N(x)$. Using the symmetry heuristics we define

$$f_{N-1}(x) = \frac{N-2}{2\pi} \int_x^{a_{N-1}} (a_{N-1} - s)^{-\frac{1}{2}} \prod_{j=1, j \neq N-1}^N (s - a_j)^{-\frac{1}{2}} (s - x)^{\frac{N}{2}-2} ds, \quad (50)$$

now we need to pick a branch for $(s - a_N)^{-\frac{1}{2}}$ for $s < a_N$. The requirement that $f_N(x) + f_{N-1}(x)$ be real for $a_{N-2} < x < a_{N-1}$ motivates the following:

1. For $0 \leq j < N$ and $a_1 < x \leq a_{N-j}$ let

$$f_{N-j}(x) = \frac{N-2}{2\pi} i^j \int_x^{a_{N-j}} \left(\prod_{m=0}^j (a_{N-m} - s)^{-\frac{1}{2}} \times \prod_{m=j+1}^{N-1} (s - a_{N-m})^{-\frac{1}{2}} (s - x)^{\frac{N}{2}-2} \right) ds, \quad (51)$$

2. for $0 \leq j \leq N-2$ and $a_{N-j-1} \leq x < a_{N-j}$ the density is

$$f(x) = \sum_{m=0}^j f_{N-m}(x), \quad (52)$$

3. if $s < a_m$ then $(s - a_m)^{-\frac{1}{2}} = -i(a_m - s)^{-\frac{1}{2}}$ for $2 \leq m \leq N-1$.

Suppose $a_1 < x < a_{N-j-1}$ for some $j \geq 0$. As a consequence we obtain then

$$\begin{aligned}
f_{N-j}(x) + f_{N-j-1}(x) = & \frac{N-2}{2\pi} i^j \int_{a_{N-j-1}}^{a_{N-j}} \prod_{m=0}^j (a_{N-m} - s)^{-\frac{1}{2}} \prod_{m=j+1}^{N-1} (s - a_{N-m})^{-\frac{1}{2}} (s-x)^{\frac{N}{2}-2} ds \\
& + \frac{N-2}{2\pi} i^j \int_x^{a_{N-j-1}} \prod_{m=0}^j (a_{N-m} - s)^{-\frac{1}{2}} \prod_{m=j+1}^{N-1} (s - a_{N-m})^{-\frac{1}{2}} (s-x)^{\frac{N}{2}-2} ds \\
& + \frac{N-2}{2\pi} i^{j+1} \int_x^{a_{N-j-1}} \prod_{m=0}^{j+1} (a_{N-m} - s)^{-\frac{1}{2}} \prod_{m=j+2}^{N-1} (s - a_{N-m})^{-\frac{1}{2}} (s-x)^{\frac{N}{2}-2} ds.
\end{aligned} \tag{53}$$

Due to equation (5) the factor $(s - a_{N-j-1})^{-\frac{1}{2}}$ in the second integral is replaced by $-i(a_{N-j-1} - s)^{-\frac{1}{2}}$. Therefore the second and third integrals cancel out as $(-i)i^j + i^{j+1} = 0$. Hence there are two different types of expressions for the density, depending on whether $a_{N-2M} < x < a_{N-2M+1}$ or $a_{N-2M-1} < x < a_{N-2M}$. For $0 \leq j \leq \lfloor \frac{N-2}{2} \rfloor$ let

$$g_j(s) := \prod_{m=0}^{2j} (a_{N-m} - s)^{-\frac{1}{2}} \prod_{m=2j+1}^{N-1} (s - a_{N-m})^{-\frac{1}{2}}, \tag{54}$$

then for $a_{N-2M} \leq x < a_{N-2M+1}$ (with $1 \leq M \leq \frac{N-1}{2}$)

$$f(x) = \frac{N-2}{2\pi} \sum_{j=0}^{M-1} (-1)^j \int_{a_{N-2j-1}}^{a_{N-2j}} g_j(s) (s-x)^{\frac{N}{2}-2} ds, \tag{55}$$

and for $a_{N-2M-1} \leq x < a_{N-2M}$ (with $0 \leq M \leq \frac{N-2}{2}$)

$$\begin{aligned}
f(x) = & \frac{N-2}{2\pi} \sum_{j=0}^{M-1} (-1)^j \int_{a_{N-2j-1}}^{a_{N-2j}} g_j(s) (s-x)^{\frac{N}{2}-2} ds \\
& + (-1)^M \frac{N-2}{2\pi} \int_x^{a_{N-2M}} g_M(s) (s-x)^{\frac{N}{2}-2} ds.
\end{aligned} \tag{56}$$

An important consequence of this formulation is that for even N the density is a polynomial of degree $\frac{N}{2} - 2$ on the *even* intervals (a_{N-2M}, a_{N-2M+1}) , which means that the parity by counting intervals from the top down is even, so that (a_{N-1}, a_N) is $\#1$.

Now we are in position to formulate the main result of this work.

Theorem 14. *For $N > 2$ the formulas (55) and (56) give the real shadow of a real symmetric matrix with spectrum $\{a_i\}_{i=1}^N$ - the density of $\mathcal{P}_{\text{diag}(a_1, \dots, a_N)}^{\mathbb{R}} = \mathcal{D}(a_1, \dots, a_N; \frac{1}{2}, \dots, \frac{1}{2})$.*

We prove the validity of the above theorem by showing that

$$\int_{a_1}^{a_N} (1 - r(x - a_1))^{-\frac{N}{2}} f(x) dx = \prod_{j=2}^N (1 - r(a_j - a_1))^{-\frac{1}{2}}, |r| < \frac{1}{a_N - a_1}, \quad (57)$$

this is the moment generating function, see Lemma 3. Start by expressing $\int_{a_1}^{a_N} (x - a_1)^n f(x) dx$ as a sum of integrals, for $n = 0, 1, 2, \dots$. The contribution of an “even” interval $a_{N-2M} \leq x \leq a_{N-2M+1}$ to the moment is

$$\frac{N-2}{2\pi} \sum_{j=0}^{M-1} (-1)^j \int_{a_{N-2M}}^{a_{N-2M+1}} (x - a_1)^n dx \int_{a_{N-2j-1}}^{a_{N-2j}} g_j(s) (s - x)^{\frac{N}{2}-2} ds, \quad (58)$$

and the contribution of an “odd” interval $a_{N-2M-1} \leq x \leq a_{N-2M}$ is

$$\begin{aligned} & \frac{N-2}{2\pi} \sum_{j=0}^{M-1} (-1)^j \int_{a_{N-2M-1}}^{a_{N-2M}} (x - a_1)^n dx \int_{a_{N-2j-1}}^{a_{N-2j}} g_j(s) (s - x)^{\frac{N}{2}-2} ds \\ & + (-1)^M \frac{N-2}{2\pi} \int_{a_{N-2M-1}}^{a_{N-2M}} (x - a_1)^n dx \int_x^{a_{N-2M}} g_M(s) (s - x)^{\frac{N}{2}-2} ds. \end{aligned} \quad (59)$$

The term $g_j(s) (s - x)^{\frac{N}{2}-2}$ appears in the intervals $a_{N-2M} \leq x \leq a_{N-2M+1}$ for $M \geq j+1$ and in $a_{N-2M-1} \leq x \leq a_{N-2M}$ for $M \geq j$. Collect these terms:

$$\begin{aligned} & \frac{N-2}{2\pi} (-1)^j \int_{a_{N-2j-1}}^{a_{N-2j}} g_{2j}(s) \left\{ \sum_{M=j+1}^{\lfloor \frac{N-1}{2} \rfloor} \int_{a_{N-2M}}^{a_{N-2M+1}} + \sum_{M=j+1}^{\lfloor \frac{N-2}{2} \rfloor} \int_{a_{N-2M-1}}^{a_{N-2M}} \right\} (x - a_1)^n (s - x)^{\frac{N}{2}-2} dx ds \\ & + \frac{N-2}{2\pi} (-1)^j \int_{a_{N-2j-1}}^{a_{N-2j}} (x - a_1)^n dx \int_x^{a_{N-2j}} g_j(s) (s - x)^{\frac{N}{2}-2} ds. \end{aligned} \quad (60)$$

The terms in the first line add up to just one interval of integration $a_1 \leq x \leq a_{N-2j-1}$. In the second line reverse the order of integration (note the region for the double integral is $a_{N-2j-1} \leq x \leq s \leq a_{N-2j}$) to obtain

$$\frac{N-2}{2\pi} (-1)^j \int_{a_{N-2j-1}}^{a_{N-2j}} g_j(s) ds \int_{a_{N-2j-1}}^s (x - a_1)^n (s - x)^{\frac{N}{2}-2} dx. \quad (61)$$

The terms with g_j add up to

$$\begin{aligned} & \frac{N-2}{2\pi} (-1)^j \int_{a_{N-2j-1}}^{a_{N-2j}} g_j(s) ds \int_{a_1}^s (x - a_1)^n (s - x)^{\frac{N}{2}-2} dx \\ & = \frac{(-1)^j}{\pi} \left(\frac{N-2}{2} \right) B \left(\frac{N}{2} - 1, n+1 \right) \int_{a_{N-2j-1}}^{a_{N-2j}} g_j(s) (s - a_1)^{\frac{N}{2}+n-1} ds, \end{aligned} \quad (62)$$

from the Beta integral $\int_a^b (b-x)^{\alpha-1} (x-a)^{\beta-1} dx = (b-a)^{\alpha+\beta-1} B(\alpha, \beta)$ with $a = a_1, b = s, \alpha = \frac{N}{2} - 1, \beta = n + 1$. Furthermore $(\frac{N-2}{2}) B(\frac{N}{2} - 1, n + 1) = \frac{n!}{(\frac{N}{2})_n}$. Therefore we have

$$\int_{a_1}^{a_N} (x - a_1)^n f(x) dx = \frac{1}{\pi} \frac{n!}{(\frac{N}{2})_n} \sum_{j=0}^{\lfloor \frac{N-2}{2} \rfloor} (-1)^j \int_{a_{N-2j-1}}^{a_{N-2j}} g_j(s) (s - a_1)^{\frac{N}{2} + n - 1} ds, \quad (63)$$

and

$$\begin{aligned} \int_{a_1}^{a_N} (1 - r(x - a_1))^{-\frac{N}{2}} f(x) dx &= \sum_{n=0}^{\infty} \frac{(\frac{N}{2})_n}{n!} r^n \int_{a_1}^{a_N} (x - a_1)^n f(x) dx \\ &= \frac{1}{\pi} \sum_{j=0}^{\lfloor \frac{N-2}{2} \rfloor} (-1)^j \int_{a_{N-2j-1}}^{a_{N-2j}} g_j(s) (s - a_1)^{\frac{N}{2} - 1} (1 - r(s - a_1))^{-1} ds, \end{aligned} \quad (64)$$

where the infinite sum converges for $|r| < \frac{1}{a_N - a_1}$. We will evaluate the integral by residue calculus applied to the analytic function

$$G(z) := \prod_{j=1}^N (z - a_j)^{-\frac{1}{2}} (z - a_1)^{\frac{N}{2} - 1} (1 - r(z - a_1))^{-1} \quad (65)$$

for fixed small $r > 0$ with suitable determination of the square roots. For real a, b with $a < b$ consider the analytic function $(z - a)^{-\frac{1}{2}} (z - b)^{-\frac{1}{2}}$ defined on $\mathbb{C} \setminus [a, b]$, that is, the complex plane with the interval $[a, b]$ removed. Set $z = a + r_1 e^{i\theta_1} = b + r_2 e^{i\theta_2}$, $r_1, r_2 > 0$ and $\theta_1 = \theta_2 = 0$ for z real and $z > b$, then let

$$(z - a)^{-\frac{1}{2}} (z - b)^{-\frac{1}{2}} := (r_1 r_2)^{-\frac{1}{2}} e^{-i(\theta_1 + \theta_2)/2} \quad (66)$$

and let θ_1, θ_2 vary continuously (from 0) to determine the values in the rest of the domain. This is well-defined: suppose z is real and $z < a$; approaching z from the upper half-plane θ_1, θ_2 change from 0 to π and $e^{-i(\theta_1 + \theta_2)/2}$ changes from 1 to $e^{-i\pi} = -1$, and approaching z from the lower half-plane θ_1, θ_2 change from 0 to $-\pi$ and $e^{-i(\theta_1 + \theta_2)/2}$ changes from 1 to $e^{i\pi} = -1$.

Lemma 15. *Suppose h is analytic in a complex neighborhood of $[a, b]$ and γ_ε is a closed contour oriented clockwise (negatively) made up of the segments $\{x + i\varepsilon : a \leq x \leq b\}$, $\{x - i\varepsilon : a \leq x \leq b\}$ and semicircles $\{a + \varepsilon e^{i\theta} : \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\}$, $\{b + \varepsilon e^{i\theta} : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$ (for sufficiently small $\varepsilon > 0$) then*

$$\lim_{\varepsilon \rightarrow 0+} \oint_{\gamma_\varepsilon} h(z) (z - a)^{-\frac{1}{2}} (z - b)^{-\frac{1}{2}} dz = -2i \int_a^b h(x) ((b - x)(a - x))^{-\frac{1}{2}} dx. \quad (67)$$

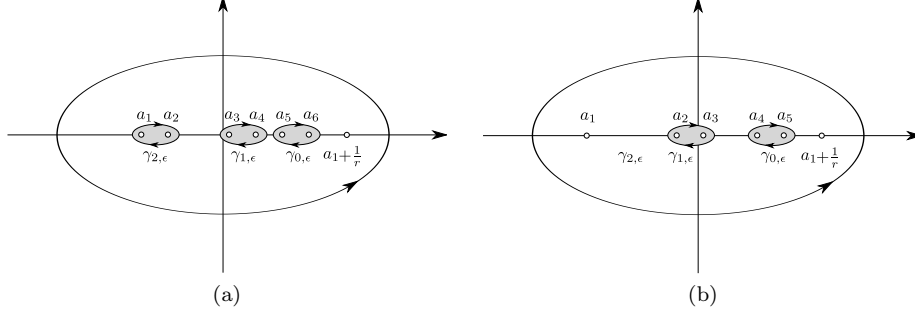


Figure 2: Visualization of the integration of the function $G(z)$ defined in (65). Panel (a) even N (here $N = 6$), panel (b) odd N (here $N = 5$).

Proof. On the semicircles the integrand is bounded by $M\varepsilon^{-\frac{1}{2}}$ for some $M < \infty$ and the length of the arc is $\pi\varepsilon$ so this part of the integral tends to zero as $\varepsilon \rightarrow 0_+$. Along $\{z = x + i\varepsilon : a \leq x \leq b\}$ the arguments are $\theta_1 \approx \pi$ and $\theta_2 \approx 0$ so $(z - a)^{-\frac{1}{2}}(z - b)^{-\frac{1}{2}} \approx e^{-i\pi/2}(r_1 r_2)^{-\frac{1}{2}}$ and this part of the integral $\approx -i \int_a^b h(x + i\varepsilon)((b - x)(a - x))^{-\frac{1}{2}} dx$. Along $\{z = x - i\varepsilon : a \leq x \leq b\}$ the arguments are $\theta_1 \approx -\pi$ and $\theta_2 \approx 0$ so $(z - a)^{-\frac{1}{2}}(z - b)^{-\frac{1}{2}} \approx e^{i\pi/2}(r_1 r_2)^{-\frac{1}{2}}$ and this part of the integral $\approx i \int_b^a h(x - i\varepsilon)((b - x)(a - x))^{-\frac{1}{2}} dx$. Adding the two pieces and letting $\varepsilon \rightarrow 0_+$ proves the claim. ■

Now fix $r > 0$ with $\frac{1}{r} > \max(|a_1|, |a_N|, a_N - a_1)$. Define a positively oriented closed contour Γ consisting of a large circle $\gamma = \{z = Re^{i\theta} : 0 \leq \theta \leq 2\pi\}$ with $R > \frac{1}{r}$ and $\{\gamma_{j,\varepsilon} : 0 \leq j \leq \lfloor \frac{N-2}{2} \rfloor\}$ where $\gamma_{j,\varepsilon}$ is a closed negatively oriented contour around the interval $[a_{N-2j-1}, a_{N-2j}]$ as in the Lemma, with $\varepsilon > 0$ sufficiently small so that the contours do not intersect – see Fig. 2. The function G is meromorphic on $\mathbb{C} \setminus \bigcup_{j=0}^{\lfloor (N-2)/2 \rfloor} [a_{N-2j-1}, a_{N-2j}]$ and has one simple pole at $z = a_1 + \frac{1}{r}$. By the (generalized) residue theorem

$$\frac{1}{2\pi i} \oint_{\Gamma} G(z) dz = \text{res}_{z=a_1+\frac{1}{r}} G(z). \quad (68)$$

Using the determinations of roots described above let $z = a_j + r_j e^{i\theta_j}$ for $1 \leq j \leq N$ with $r_j > 0$. For large $|z|$ we see $|G(z)| < M|z|^{-2}$ so the integral around γ (circle with radius R) tends to zero as $R \rightarrow \infty$. Consider N even or odd separately.

5.1. Case of odd N :

The interval with the lowest index is $[a_2, a_3]$ and the analytic function $G(z) = \prod_{j=2}^N (z - a_j)^{-\frac{1}{2}} (z - a_1)^{\frac{N-3}{2}} (1 - r(z - a_1))^{-1}$. Here $\frac{N-3}{2}$ is an integer

thus $(z - a_1)^{\frac{N-3}{2}}$ is entire. Applying the Lemma to $\gamma_{j,\varepsilon}$ put

$$h(z) = \prod_{m=0}^{2j-1} (z - a_{N-m})^{-\frac{1}{2}} \prod_{m=2j+2}^{N-2} (z - a_{N-m})^{-\frac{1}{2}} (z - a_1)^{\frac{N-3}{2}} (1 - r(z - a_1))^{-1}. \quad (69)$$

In this case $\theta_m = \pi$ for $N - 2j + 1 \leq m \leq N$ and $\theta_m = 0$ for $1 \leq m \leq N - 2j - 2$ so for $a_{N-2j-1} \leq x \leq a_{N-2j}$ we have

$$\begin{aligned} h(x) &= e^{-i(2j\pi)/2} \prod_{m=0}^{2j-1} (a_{N-m} - x)^{-\frac{1}{2}} \times \\ &\times \prod_{m=2j+2}^{N-2} (x - a_{N-m})^{-\frac{1}{2}} (x - a_1)^{\frac{N-3}{2}} (1 - r(x - a_1))^{-1} \end{aligned} \quad (70)$$

and

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0_+} \oint_{\gamma_{j,\varepsilon}} G(z) dz &= -2i(-1)^j \int_{a_{N-2j-1}}^{a_{N-2j}} \left(\prod_{m=0}^{2j} (a_{N-m} - x)^{-\frac{1}{2}} \right. \\ &\left. \prod_{m=2j+1}^{N-2} (x - a_{N-m})^{-\frac{1}{2}} (x - a_1)^{\frac{N-3}{2}} (1 - r(x - a_1))^{-1} \right) dx. \end{aligned} \quad (71)$$

The residue at $z = a_1 + \frac{1}{r}$ is straightforward:

$$\begin{aligned} \lim_{z \rightarrow a_1 + \frac{1}{r}} \left(z - a_1 - \frac{1}{r} \right) G(z) &= -\frac{1}{r} \prod_{j=2}^N \left(\frac{1}{r} - (a_j - a_1) \right)^{-\frac{1}{2}} \left(\frac{1}{r} \right)^{\frac{N-3}{2}} \\ &= -\prod_{j=2}^N (1 - r(a_j - a_1))^{-\frac{1}{2}}, \end{aligned} \quad (72)$$

because $\frac{1}{r} > a_N$ and the determination of the roots gives positive values. Thus in the limit as $\varepsilon \rightarrow 0_+, R \rightarrow \infty$ we obtain

$$\begin{aligned} \frac{1}{2\pi i} \sum_{j=0}^{\frac{N-3}{2}} (-1)^j (-2i) \int_{a_{N-2j-1}}^{a_{N-2j}} g_{2j}(x) (x - a_1)^{\frac{N}{2}-1} (1 - r(x - a_1))^{-1} dx = \\ - \prod_{j=2}^N (1 - r(a_j - a_1))^{-\frac{1}{2}}, \end{aligned} \quad (73)$$

and this is the required result.

5.2. Case of N even

The interval with the lowest index is $[a_1, a_2]$ and the function $G(z) = \prod_{j=1}^N (z - a_j)^{-\frac{1}{2}} (z - a_1)^{\frac{N-2}{2}} (1 - r(z - a_1))^{-1}$. Here $\frac{N-2}{2}$ is an integer so $(z - a_1)^{\frac{N-2}{2}}$

is entire. Applying the Lemma to $\gamma_{j,\varepsilon}$ put

$$h(z) = \prod_{m=0}^{2j-1} (z - a_{N-m})^{-\frac{1}{2}} \prod_{m=2j+2}^{N-1} (z - a_{N-m})^{-\frac{1}{2}} (z - a_1)^{\frac{N-2}{2}} (1 - r(z - a_1))^{-1}. \quad (74)$$

In this case $\theta_m = \pi$ for $N - 2j + 1 \leq m \leq N$ and $\theta_m = 0$ for $1 \leq m \leq N - 2j - 2$ so for $a_{N-2j-1} \leq x \leq a_{N-2j}$ we have

$$\begin{aligned} h(x) = & e^{-i(2j\pi)/2} \prod_{m=0}^{2j-1} (a_{N-m} - x)^{-\frac{1}{2}} \times \\ & \times \prod_{m=2j+2}^{N-1} (x - a_{N-m})^{-\frac{1}{2}} (x - a_1)^{\frac{N-2}{2}} (1 - r(x - a_1))^{-1} \end{aligned} \quad (75)$$

and

$$\begin{aligned} \lim_{\substack{\varepsilon \rightarrow 0_+ \\ \gamma_{j,\varepsilon}}} \oint G(z) dz = & -2i(-1)^j \int_{a_{N-2j-1}}^{a_{N-2j}} \left(\prod_{m=0}^{2j} (a_{N-m} - x)^{-\frac{1}{2}} \times \right. \\ & \times \left. \prod_{m=2j+1}^{N-1} (x - a_{N-m})^{-\frac{1}{2}} (x - a_1)^{\frac{N-2}{2}} (1 - r(x - a_1))^{-1} \right) dx. \end{aligned} \quad (76)$$

The residue at $z = a_1 + \frac{1}{r}$ is:

$$\begin{aligned} \lim_{z \rightarrow a_1 + \frac{1}{r}} \left(z - a_1 - \frac{1}{r} \right) G(z) = & -\frac{1}{r} \prod_{j=1}^N \left(\frac{1}{r} - (a_j - a_1) \right)^{-\frac{1}{2}} \left(\frac{1}{r} \right)^{\frac{N-2}{2}} \\ = & -\prod_{j=2}^N (1 - r(a_j - a_1))^{-\frac{1}{2}}, \end{aligned} \quad (77)$$

because $\frac{1}{r} > a_N$ and the determination of the roots gives positive values. Thus in the limit as $\varepsilon \rightarrow 0_+, R \rightarrow \infty$ we obtain the final result

$$\begin{aligned} \frac{1}{2\pi i} \sum_{j=0}^{\frac{N-2}{2}} (-1)^j (-2i) \int_{a_{N-2j-1}}^{a_{N-2j}} g_{2j}(x) (x - a_1)^{\frac{N}{2}-1} (1 - r(x - a_1))^{-1} ds = \\ - \prod_{j=2}^N (1 - r(a_j - a_1))^{-\frac{1}{2}}. \end{aligned} \quad (78)$$

For distributions supported by bounded intervals the moment generating function determines the distribution uniquely. Thus we have established the Theorem 14.

5.3. Examples

There is a somewhat disguised complete elliptic integral of the first kind which appears in $N = 3, 4, 5$. For $b_1 < b_2 < b_3 < b_4$ let

$$E(b_1, b_2; b_3, b_4) := \frac{1}{\pi} \int_{b_3}^{b_4} \{(b_4 - s)(s - b_3)(s - b_2)(s - b_1)\}^{-\frac{1}{2}} ds. \quad (79)$$

There is a hypergeometric formulation (see formula (25) with $k = \frac{1}{2}$):

$$E(b_1, b_2; b_3, b_4) := \frac{1}{\sqrt{(b_3 - b_1)(b_4 - b_2)}} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{(b_4 - b_3)(b_2 - b_1)}{(b_3 - b_1)(b_4 - b_2)}\right). \quad (80)$$

Consider the density for $N = 3$ and $a_1 < a_2 < x < a_3$; by formula (56)

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_x^{a_3} (a_3 - s)^{-\frac{1}{2}} \prod_{j=1}^2 (s - a_j)^{-\frac{1}{2}} (s - x)^{-\frac{1}{2}} ds \\ &= \frac{1}{2} E(a_1, a_2; x, a_3). \end{aligned} \quad (81)$$

Similarly formula (55) shows that $f(x) = \frac{1}{2} E(a_1, x; a_2, a_3)$ for $a_1 < x < a_2$.

Suppose $N = 4$ and $a_2 < x \leq a_3$ then by formula (55)

$$\begin{aligned} f(x) &= \frac{1}{\pi} \int_{a_3}^{a_4} (a_4 - s)^{-\frac{1}{2}} \prod_{j=1}^3 (s - a_j)^{-\frac{1}{2}} ds = f(a_3) \\ &= E(a_1, a_2; a_3, a_4) \end{aligned} \quad (82)$$

and the density is constant on this interval.

Suppose $N = 5$ and $a_3 \leq x < a_4$ then by formula (55)

$$f(x) = \frac{3}{2\pi} \int_{a_4}^{a_5} (a_5 - s)^{-\frac{1}{2}} \prod_{j=1}^4 (s - a_j)^{-\frac{1}{2}} (s - x)^{\frac{1}{2}} ds, \quad (83)$$

$$\begin{aligned} f(a_3) &= \frac{3}{2\pi} \int_{a_4}^{a_5} (a_5 - s)^{-\frac{1}{2}} (s - a_4)^{-\frac{1}{2}} \prod_{j=1}^2 (s - a_j)^{-\frac{1}{2}} ds \\ &= \frac{3}{2} E(a_1, a_2; a_4, a_5), \end{aligned} \quad (84)$$

which is independent of a_3 .

The integrals in the density formula have the form $\frac{1}{\pi} \int_a^b h(s) ((b-s)(s-a))^{-\frac{1}{2}} ds$, where h is differentiable in a neighborhood of $[a, b]$. The technique of Gauss-Chebyshev quadrature is well suited for the numerical evaluation of the desired densities: Set $\phi(t) = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)t$ then the sums $\frac{1}{n} \sum_{j=0}^{n-1} h\left(\phi\left(\cos \frac{(2j+1)\pi}{2n}\right)\right)$ converge rapidly to the integral (as $n \rightarrow \infty$); typically $n = 20$ suffices for reasonable accuracy.

Another way of numerical approximation of a real numerical shadow can be done by direct numerical integration of a formula for a cumulative distribution function given in [15].

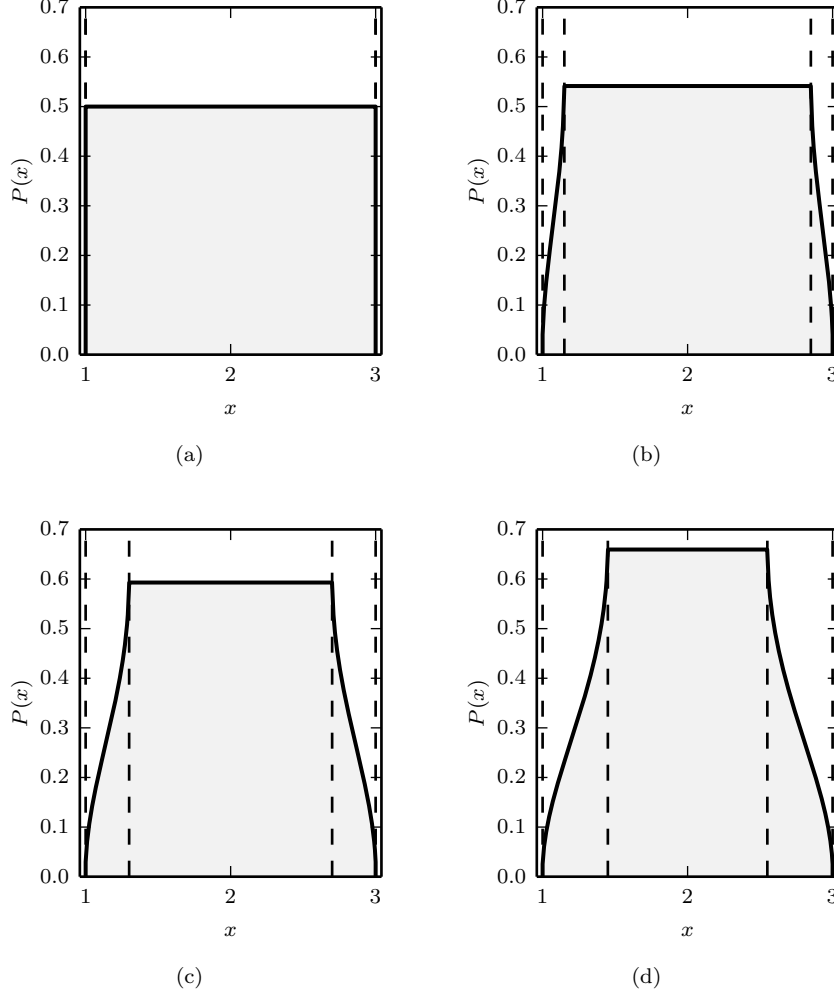


Figure 3: Real numerical shadow $\mathcal{P}_A^{\mathbb{R}}(x)$ of a diagonal matrix $A = \text{diag}(1, 1 + \epsilon, 3 - \epsilon, 3)$ of order $N = 4$, where (a) $\epsilon = 0$ (degenerated case: the real shadow is equivalent to the complex shadow of the reduced matrix, $\text{Tr}_2 A = \text{diag}(1, 3)$); (b) $\epsilon = 0.15$; (c) $\epsilon = 0.3$ and (d) $\epsilon = 0.45$. Note that for $x \in (a_2, a_3)$ all distributions are flat.

6. Continuity at the knots

In this section we examine the behavior of the shadow density at the knots, where the curve pieces meet, that is, the regular singular points of the shadow differential equation. The even and odd N cases are quite different. For even N there are even and odd segments based on counting from a_N , so $[a_{N-1}, a_N]$

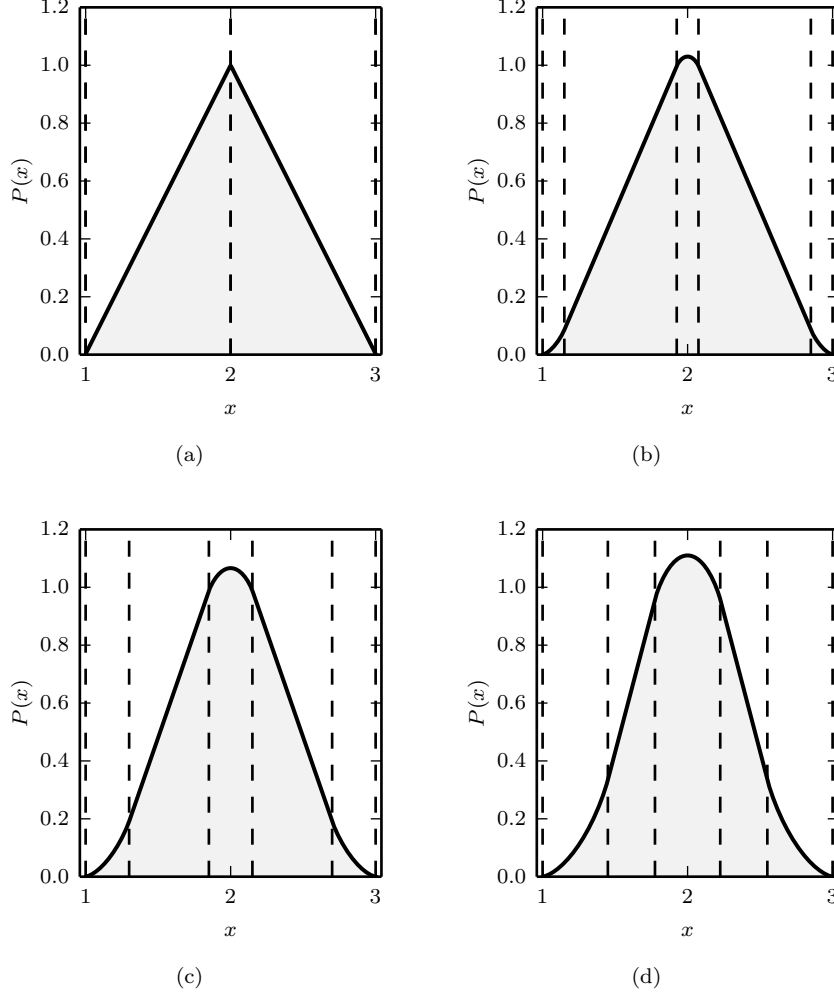


Figure 4: Real numerical shadow of matrix A of order $N = 6$ $A = \text{diag}(1, 1 + \epsilon, 2 - \epsilon/2, 2 + \epsilon/2, 3 - \epsilon, 3)$, where (a) $\epsilon = 0$ (degenerated case: the real shadow is equivalent to the complex shadow of the reduced matrix, $\text{Tr}_2 A = \text{diag}(1, 2, 3)$); (b) $\epsilon = 0.15$; (c) $\epsilon = 0.3$; (d) $\epsilon = 0.45$. Note that for $x \in (a_2, a_3)$ and $x \in (a_4, a_5)$ the distributions are linear.

is #1, and this parity is the same if one counts up from a_1 . For odd N there are even and odd knots (the parity of j for the knot a_j ; this remains the same under the transformation $x \mapsto -x$). In the neighborhood of each knot a_j there is the analytic part, expandable in a power series $\sum_{n=0}^{\infty} c_n (x - a_j)^n$, and a part with discontinuous derivative of order $\lfloor \frac{N}{2} \rfloor - 1$, as will be shown. For

even $N = 2M$ the density is polynomial of degree $M - 2$ in x in the even intervals $[a_{N-2j}, a_{N-2j+1}]$, and has a jump of the form $|x - a_i|^{M-\frac{3}{2}}$ at each end on the odd intervals $[a_{N-2j-1}, a_{N-2j}]$. Recall that the critical exponents are $\frac{N-1}{2} - 1 = M - \frac{3}{2}$ and $0, 1, \dots, N - 3$.

For odd $N = 2M + 1$ there is just one type of curve piece: behavior like $|x - a_i|^{M-1}$ at the odd end-point and $|x - a_i|^{M-1} \log |x - a_i|$ at the even end-point. The critical exponent $M - 1 = \frac{N-1}{2} - 1$ is repeated, accounting for the logarithmic term. In this case each interval can be considered as even or odd by starting from a_N or from a_1 and using the transformation $x \mapsto -x$.

For $a_{N-2m-1} \leq x < a_{N-2m}$ we have

$$\begin{aligned} f(x) &= \frac{N-2}{2\pi} \sum_{j=0}^{m-1} (-1)^j \int_{a_{N-2j-1}}^{a_{N-2j}} g_j(s) (s-x)^{\frac{N}{2}-2} ds \\ &\quad + \frac{N-2}{2\pi} (-1)^m \int_x^{a_{N-2m}} g_m(s) (s-x)^{\frac{N}{2}-2} ds. \end{aligned} \quad (85)$$

The integral indexed by j is analytic in $x < a_{N-2j-1}$. Furthermore if $N = 2M$ then the sum defines a polynomial in x without any further restrictions on x . Consider the integral in the second line for $a_{N-2m} - \varepsilon < x < a_{N-2m}$ for some small $\varepsilon > 0$ (and $< a_{N-2m} - a_{N-2m-1}$). Set $h_m(s) = g_m(s) (a_{N-2m} - s)^{\frac{1}{2}}$ so that $h_m(s)$ has a power series expansion $\sum_{n=0}^{\infty} c_n (a_{N-2m} - s)^n$ valid in a neighborhood of $[a_{N-2m} - r, a_{N-2m}]$ for small enough r . Then

$$\begin{aligned} \int_x^{a_{N-2m}} g_m(s) (s-x)^{\frac{N}{2}-2} ds &= \int_x^{a_{N-2m}} (a_{N-2m} - s)^{-\frac{1}{2}} (s-x)^{\frac{N}{2}-2} h_m(s) ds \\ &= \sum_{n=0}^{\infty} c_n \int_x^{a_{N-2m}} (a_{N-2m} - s)^{-\frac{1}{2}+n} (s-x)^{\frac{N}{2}-2} ds \\ &= (a_{N-2m} - x)^{\frac{N-3}{2}} \sum_{n=0}^{\infty} B\left(n + \frac{1}{2}, \frac{N-2}{2}\right) c_n (a_{N-2m} - x)^n \\ &= B\left(\frac{1}{2}, \frac{N-2}{2}\right) (a_{N-2m} - x)^{\frac{N-3}{2}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{\left(\frac{N-1}{2}\right)_n} c_n (a_{N-2m} - x)^n; \end{aligned} \quad (86)$$

this is the solution of the shadow equation for the critical exponent $\frac{N-3}{2}$ at the regular singular point a_{N-2m} . The leading term is

$$\frac{(-1)^m}{B\left(\frac{N-1}{2}, \frac{1}{2}\right)} \prod_{j=0}^{2m-1} (a_{N-j} - a_{N-2m})^{-\frac{1}{2}} \prod_{j=2m+1}^{N-1} (a_{N-2m} - a_{N-j})^{-\frac{1}{2}} (a_{N-2m} - x)^{\frac{N-3}{2}}, \quad (87)$$

in analogy to the leading term in formula (6). The computation uses an identity,

$$\frac{1}{\pi} \left(\frac{N}{2} - 1\right) \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{N-2}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)} = \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}.$$

6.1. Even N

Set $N = 2M$. Near a knot a_{2M-2m} ($0 \leq m < M$) the density $f(x)$ is polynomial for $a_{2M-2m} < x$ and given by the sum of the polynomial and a series $(a_{2M-2m} - x)^{M-\frac{3}{2}} \sum_{n=0}^{\infty} c_n (a_{2M-2m} - x)^n$ for $x < a_{2M-2m}$. Thus $f^{(j)}(x)$ is continuous in a neighborhood of a_{2M-2m} for $0 \leq j \leq M-2$. By applying this result to the reversed knots $b_1 < \dots < b_N$ where $b_j = -a_{2M+1-j}$ and x replaced by $-x$ we find that near a knot $a_{2M-2m+1} = -b_{2M-2(M-m)}$ for $1 \leq m \leq M$ the density $f(x)$ is polynomial for $x < a_{2M-2m+1}$ and is given by the sum of the polynomial and a series $(x - a_{2M-2m+1})^{M-\frac{3}{2}} \sum_{n=0}^{\infty} c_n (x - a_{2M-2m+1})^n$ for $x > a_{2M-2m+1}$. Thus the lowest order discontinuity of the density is in $f^{(M-1)}(x)$ at each knot, that is, $f^{(j)}(x)$ is continuous everywhere for all $j \leq \frac{N}{2} - 2$.

6.2. Odd N

Set $N = 2M+1$. Consider the even knot a_{N-2m-1} with $0 \leq m < M-1$. Pick r with $0 < r < \min(a_{N-2m} - a_{N-2m-1}, a_{N-2m-1} - a_{N-2m-2})$, then in the interval $[a_{N-2m-1} - r, a_{N-2m-1} + r]$ the function $\prod_{j=0}^{2m} (a_{N-j} - s)^{-\frac{1}{2}} \prod_{j=2m-2}^{N-1} (s - a_{N-j})^{-\frac{1}{2}}$ can be expanded as a power series $\sum_{n=0}^{\infty} c_n (s - a_{N-2m-1})^n$. Here the coefficients can be found by using the negative binomial theorem for each factor in the product. Since all the difficulty happens at the knot set

$$\begin{aligned} \phi(x) &= \frac{N-2}{2\pi} \sum_{j=0}^{m-1} (-1)^j \int_{a_{N-2j-1}}^{a_{N-2j}} g_j(s) (s-x)^{\frac{N}{2}-2} ds \\ &\quad + \frac{N-2}{2\pi} (-1)^m \int_{a_{N-2m-1}+r}^{a_{N-2m}} g_m(s) (s-x)^{\frac{N}{2}-2} ds. \end{aligned} \quad (88)$$

Thus $\phi(x)$ is analytic for $x < a_{N-2m-1} + r$; for $a_{N-2m-1} < x < a_{N-2m-1} + r$

$$f(x) = \phi(x) + \frac{N-2}{2\pi} (-1)^m \int_x^{a_{N-2m-1}+r} g_m(s) (s-x)^{M-\frac{3}{2}} ds, \quad (89)$$

and for $a_{N-2m-1} - r < x < a_{N-2m-1}$

$$f(x) = \phi(x) + \frac{N-2}{2\pi} (-1)^m \int_{a_{N-2m-1}}^{a_{N-2m-1}+r} g_m(s) (s-x)^{M-\frac{3}{2}} ds. \quad (90)$$

By using the power series and the change of variable $x = a_{N-2m-1} + y$ and $s = t + a_{N-2m-1}$ the first integral becomes

$$\sum_{n=0}^{\infty} c_n \int_y^r t^{n-\frac{1}{2}} (t-y)^{M-\frac{3}{2}} dt, \quad 0 < y < r, \quad (91)$$

and the second integral becomes

$$\sum_{n=0}^{\infty} c_n \int_0^r t^{n-\frac{1}{2}} (t-y)^{M-\frac{3}{2}} dt, -r < y < 0. \quad (92)$$

We want to analyze the behavior of the integrals in the limit $y \rightarrow 0$. In each integral change the variable $t = \frac{y}{1-u^2}$, so $dt = \frac{2yu}{(1-u^2)^2} du$. Furthermore $t(t-y) = \frac{y^2 u^2}{(1-u^2)^2}$ thus $\frac{1}{\sqrt{t(t-y)}} = \frac{1-u^2}{yu}$ (if $y < 0$ then $u^2 > 1$, and if $y > 0$ then $u^2 < 1$ so that this is the positive root). Set $u_r = \sqrt{\frac{r-y}{r}}$. For $0 < y < r$ the integral is

$$2y^{n+M-1} \int_0^{u_r} \frac{u^{2M-2}}{(1-u^2)^{n+M}} du, \quad 0 < u_r < 1, \quad (93)$$

and for $-r < y < 0$ the integral is

$$2y^{n+M-1} \int_{\infty}^{u_r} \frac{u^{2M-2}}{(1-u^2)^{n+M}} du, \quad u_r > 1. \quad (94)$$

Because the integrand is even we deduce that the partial fraction expansion is of the form

$$\frac{u^{2M-2}}{(1-u^2)^{n+M}} = \sum_{j=1}^{n+M} \beta_j(M, n) \left\{ \frac{1}{(1-u)^j} + \frac{1}{(1+u)^j} \right\}, \quad (95)$$

for certain constants $\beta_j(M, n)$. Thus

$$\begin{aligned} I_{M,n}(u) &:= \int \frac{u^{2M-2}}{(1-u^2)^{n+M}} du = \beta_1(M, n) \log \left| \frac{1+u}{1-u} \right| + \\ &+ \sum_{j=2}^{n+M} \frac{\beta_j(M, n)}{j-1} \left\{ \frac{1}{(1-u)^{j-1}} - \frac{1}{(1+u)^{j-1}} \right\}. \end{aligned} \quad (96)$$

This antiderivative $I_{M,n}$ vanishes at $u = 0$ and at $u = \infty$, thus both integrals have the same value $2y^{n+M-1} I_{M,n} \left(\sqrt{\frac{r-y}{r}} \right)$. The terms for $2 \leq j \leq n+M$ contribute

$$\frac{4y^{n+M-1} u_r}{(1-u_r^2)^{j-1}} \sum_{i=0}^{\lfloor (j-2)/2 \rfloor} \binom{j-1}{2i+1} u_r^{2i} = 4y^{n+M-j} \sqrt{\frac{r-y}{r}} \sum_{i=0}^{\lfloor (j-2)/2 \rfloor} \binom{j-1}{2i+1} (r-y)^i r^{j-1-i}, \quad (97)$$

which is analytic in y for $-r < y < r$. So all the singular behavior stems from the logarithmic term

$$2y^{n+M-1} \beta_1(M, n) \log \left| \frac{1+u_r}{1-u_r} \right|, \quad (98)$$

and

$$\begin{aligned} \log \left| \frac{1+u_r}{1-u_r} \right| &= -\log |1-u_r^2| + 2\log |1+u_r| \\ &= -\log |y| + \log r + 2\log \left(1 + \sqrt{\frac{r-y}{r}}\right). \end{aligned} \quad (99)$$

Collecting the relevant terms we see that for $a_{N-2m-1} - r < x < a_{N-2m-1} + r$ the density $f(x)$ is the sum of an analytic part and

$$\begin{aligned} &\frac{2}{\pi} (-1)^{m-1} (2M-1) \log |x - a_{N-2m-1}| \times \\ &\times (x - a_{N-2m-1})^{M-1} \sum_{n=0}^{\infty} \beta_1(M, n) c_n (x - a_{N-2m-1})^n. \end{aligned} \quad (100)$$

The coefficients $\beta_j(M, n)$ can be found explicitly as sums but we are only concerned with $\beta_1(M, n)$. Indeed (proof left for reader)

$$\beta_1(M, n) = \frac{1}{2} (-1)^{M-1} \frac{\left(\frac{1}{2}\right)_{M-1} \left(\frac{1}{2}\right)_n}{(n+M-1)!}. \quad (101)$$

Thus we analyzed the behavior at the even knots and showed that $f^{(j)}(x)$ is continuous everywhere for all $j \leq M-2 = \frac{N-1}{2} - 2$.

7. Entangled shadow

In previous sections we investigated the shadow with respect to real states. Here we discuss another example of the restricted shadow – the shadow with respect to maximally entangled states, briefly called *entangled shadow*.

7.1. Entangled shadow of 4×4 matrices with direct sum structure

We shall start recalling the definition of the entangled shadow introduced in [11].

Definition 16. *Maximally entangled numerical shadow of a matrix A of size $N = N_1 \times N_2$ is defined as a probability distribution $\mathcal{P}_A^{\text{ent}}(z)$ on the complex plane.*

$$\mathcal{P}_A^{\text{ent}}(z) := \int d\mu(\psi) \delta\left(z - \langle \psi | A | \psi \rangle\right), \quad (102)$$

where $\mu(\psi)$ denotes the unique unitarily invariant (Fubini-Study) measure on the set of complex pure states restricted to the set of bi-partite maximally entangled states

$$\left\{ |\psi\rangle \in \mathbb{C}^{N_1 \times N_2} : |\psi\rangle = \frac{1}{\sqrt{N_{\min}}} (U_1 \otimes U_2) \sum_{i=1}^{N_{\min}} |\psi_i^1\rangle \otimes |\psi_i^2\rangle \right\}. \quad (103)$$

Here $N_{\min} = \min(N_1, N_2)$, while $|\psi_i^1\rangle, |\psi_i^2\rangle$ form orthonormal bases in \mathbb{C}^{N_1} and \mathbb{C}^{N_2} respectively, while $U_1 \in SU(N_1)$ and $U_2 \in SU(N_2)$.

Definition 17. Pauli matrices σ_x , σ_y and σ_z are defined as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (104)$$

Lemma 18. For a 2×2 unitary matrix U and an arbitrary 2×2 matrix A we have

$$\langle 1|U^\dagger A U|1\rangle = \langle 0|U^\dagger \sigma_y A^\top \sigma_y U|0\rangle, \quad (105)$$

Theorem 19. Maximally entangled shadow

$$\mathcal{P}_{A \oplus B}^{\text{ent}} = \mathcal{P}_{\frac{1}{2}(A + \sigma_y B^\top \sigma_y)}, \quad (106)$$

where $A \oplus B$ denotes block matrix.

Proof. We write

$$\begin{aligned} \langle \psi|(A \oplus B)|\psi\rangle &= \langle \psi_+|(\mathbf{1} \otimes U^\dagger)(A \oplus B)(\mathbf{1} \otimes U)|\psi_+\rangle \\ &= \langle \psi_+|(U^\dagger \oplus U^\dagger)(A \oplus B)(U \oplus U)|\psi_+\rangle \\ &= \frac{1}{2} (\langle 0|U^\dagger A U|0\rangle + \langle 1|U^\dagger B U|1\rangle). \end{aligned} \quad (107)$$

Now we use lemma and write

$$\begin{aligned} \langle \psi|(A \oplus B)|\psi\rangle &= \frac{1}{2} (\langle 0|U^\dagger A U|0\rangle + \langle 1|U^\dagger B U|1\rangle) \\ &= \frac{1}{2} (\langle 0|U^\dagger A U|0\rangle + \langle 0|U^\dagger \sigma_y B^\top \sigma_y U|0\rangle) \\ &= \frac{1}{2} (\langle 0|U^\dagger (A + \sigma_y B^\top \sigma_y) U|0\rangle). \end{aligned} \quad (108)$$

■

This theorem is valid for complex and real entangled shadow. Also we made no assumptions on A and B , hence it is valid for non-normal matrices.

7.2. Real maximally entangled shadow of 4×4 matrices

Definition 20. Real maximally entangled numerical shadow $\mathcal{P}_A^{\text{ent}|\mathbb{R}}$ of a matrix A of size $N = N_1 \times N_2$ is defined similarly to the maximally entangled shadow, but with restriction to the real maximally entangled states.

The following theorem gives a full characterization of the real maximally entangled numerical shadow of 4×4 matrices.

Theorem 21. Let A be any 4×4 matrix then we have

$$\mathcal{P}_A^{\text{ent}|\mathbb{R}} = \frac{1}{2} \mathcal{P}_{Z_1^\top A Z_1}^{\mu_{\mathbb{R}}} + \frac{1}{2} \mathcal{P}_{Z_2^\top A Z_2}^{\mu_{\mathbb{R}}}, \quad (109)$$

where, Z_1 and Z_2 are

$$Z_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Z_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (110)$$

Proof. Any real maximally entangled pure state $|\psi\rangle$ of size four may be written as a vector obtained from the elements of an orthogonal matrix of order two,

$$|\psi\rangle = \frac{1}{\sqrt{2}} \text{vec}(O(\theta)). \quad (111)$$

First we consider an orthogonal matrix $O(\theta)$ satisfying $\det O(\theta) = 1$. We have

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos \theta \\ \sin \theta \\ -\sin \theta \\ \cos \theta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}. \quad (112)$$

Hence,

$$\langle \psi | A | \psi \rangle = \langle r | Z_1^T A Z_1 | r \rangle. \quad (113)$$

Now we consider an orthogonal matrix $O(\theta)$ satisfying $\det O(\theta) = -1$. We have

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos \theta \\ \sin \theta \\ \sin \theta \\ -\cos \theta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}. \quad (114)$$

Hence,

$$\langle \psi | A | \psi \rangle = \langle r | Z_2^T A Z_2 | r \rangle. \quad (115)$$

Combining equations (113) and (115) we get the theorem. ■

Examples are shown in Figures 5a and 5b. The theorem is valid for non-normal matrices.

7.3. Complex maximally entangled shadow of 4×4 matrices

Theorem 22. Given an arbitrary matrix A of order four its complex maximally entangled shadow is equal to the real shadow of matrix $W^\dagger A W$

$$\mathcal{P}_A^{\mu_{\text{ent}}} = \mathcal{P}_{W^\dagger A W}^{\mu_{\mathbb{R}}}, \quad (116)$$

where W is the matrix representing the 'magic basis',

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & i \\ -1 & i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & 1 & -i \end{pmatrix}. \quad (117)$$

The above theorem is related to the well known fact in the group theory, that

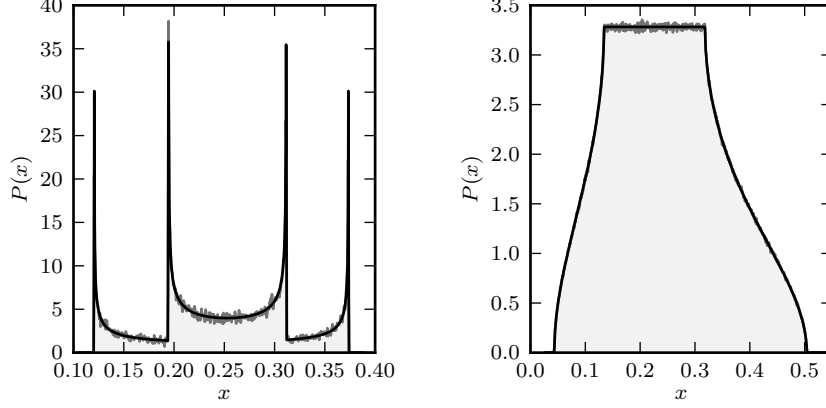
$$SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2. \quad (118)$$

Proof. Any maximally entangled two-qubit state $|\psi\rangle$ can be written as

$$|\psi\rangle = \text{vec}(V), \text{ where } V \in SU(2). \quad (119)$$

Using a parameterization of $SU(2)$ we can write

$$V = \begin{pmatrix} e^{i\xi_2} \cos \eta & e^{i\xi_1} \sin \eta \\ -e^{-i\xi_1} \sin \eta & e^{-i\xi_2} \cos \eta \end{pmatrix}. \quad (120)$$



(a) Real maximally entangled shadow (b) Complex maximally entangled shadow

Figure 5: Fig a: Real maximally entangled numerical shadow of a Hermitian matrix sampled from the Hilbert-Schmidt distribution. Monte Carlo integration result (light gray) and shadow composition based on Theorem 21 (black). Fig. b: complex maximally entangled numerical shadow of a generic Hermitian matrix. Monte Carlo integration result (light gray) and shadow composition based on Theorem 22 (black). The difference in heights of the peaks (case a) and fluctuations at the central segment of the spectrum (case b) are due to numerical errors.

Reshaping this matrix into a vector of length four we obtain the state

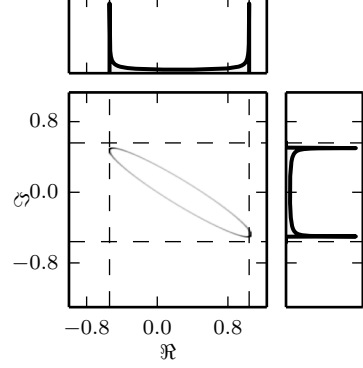
$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos \xi_2 \sin \eta + i \sin \xi_2 \cos \eta \\ -\cos \xi_1 \sin \eta + i \sin \xi_1 \sin \eta \\ \cos \xi_1 \sin \eta + i \sin \xi_1 \sin \eta \\ \cos \xi_2 \cos \eta - i \sin \xi_2 \cos \eta \end{pmatrix}. \quad (121)$$

On the other hand, consider the Hopf parameterization of the 3-sphere S^3 embedded in \mathbb{C}^2 . A point on this sphere can be expressed as

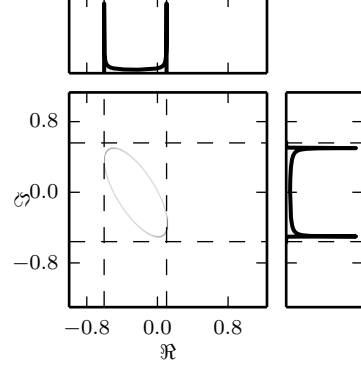
$$\begin{cases} z_1 = e^{i\xi_1} \sin \eta \\ z_2 = e^{i\xi_2} \cos \eta. \end{cases} \quad (122)$$

A point on the 3-sphere may be written in real coordinates (r_1, r_2, r_3, r_4) as

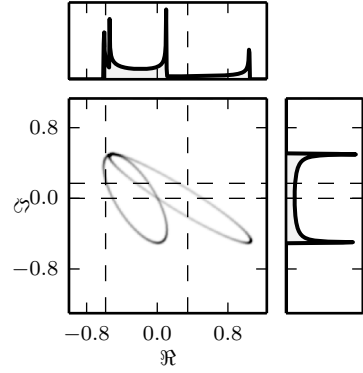
$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} \Re z_1 \\ \Im z_1 \\ \Re z_2 \\ \Im z_2 \end{pmatrix} = \begin{pmatrix} \cos \xi_1 \sin \eta \\ \sin \xi_1 \sin \eta \\ \cos \xi_2 \cos \eta \\ \sin \xi_2 \cos \eta \end{pmatrix}. \quad (123)$$



(a) Real numerical shadow of matrix $Z_1AZ_1^\dagger$ obtained using Monte-Carlo sampling.



(b) Real numerical shadow of matrix $Z_2AZ_2^\dagger$ obtained using Monte-Carlo sampling.



(c) Real entangled numerical shadow of matrix A using Monte-Carlo sampling.

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & i & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

Figure 6: Visualisation of Theorem 21 using matrix A .

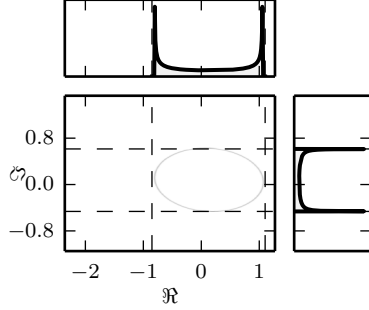
Now, using Equation (123), we can rewrite Equation (121) as

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} r_3 + ir_4 \\ -r_1 + ir_2 \\ r_1 + ir_2 \\ r_3 - ir_4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & i \\ -1 & i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & 1 & -i \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix}. \quad (124)$$

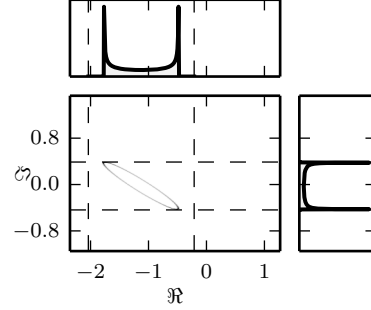
Hence, we can write

$$\langle\psi|A|\psi\rangle = \langle r|W^\dagger AW|r\rangle, \quad (125)$$

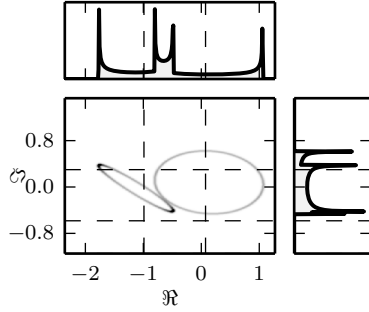
where $|r\rangle$ is a real vector defined in equation (123). ■



(a) Real numerical shadow of matrix $Z_1 B Z_1^\dagger$ obtained using Monte-Carlo sampling.



(b) Real numerical shadow of matrix $Z_2 B Z_2^\dagger$ obtained using Monte-Carlo sampling.



(c) Real entangled numerical shadow of matrix B using Monte-Carlo sampling.

$$B = \begin{pmatrix} 0.3+0.5i & -0.8-0.2i & 0.4-0.5i & 1 \\ 0.6-0.8i & -0.8-0.4i & -0.6+0.8i & -0.8+0.8i \\ 0.7-0.8i & -0.5-0.4i & -0.8 & 0.7-0.3i \\ 0.4+0.6i & -1.-0.8i & -0.4-0.4i & -0.7 \end{pmatrix}$$

Figure 7: Visualisation of Theorem 21 using matrix B .

8. Concluding remarks

In this work we analyzed probability distributions on the complex plane induced by projecting the set of quantum states, (i.e. Hermitian, positive and normalized matrices of a given size N), endowed with a certain probability measure. In the case of the unique, unitarily invariant Haar measure on the set of complex pure states, this distribution coincides with the standard numerical shadow [6, 7] of a certain matrix A of size N . The case of a normal matrix corresponds to the projection of the unit simplex covered uniformly onto a plane [8]. If the matrix A is Hermitian, its (complex) numerical shadow is supported on an interval on the real axis, and is equivalent to the B -spline with knots at the eigenvalues of A .

The real shadow of a matrix corresponds to the Haar measure restricted to the set of real pure states [11]. For a real symmetric A its real shadow is shown to be equivalent to a to the projection of the unit simplex covered by the Dirichlet measure. The main result of this work consists in Theorem 14, which

establishes an explicit exact formula for the real shadow of any real symmetric A with prescribed spectrum (a_1, \dots, a_N) .

As the real shadow of a matrix corresponds to the Dirichlet distribution with its parameters equal to $k_1 = k_2 = \dots = k_N = 1/2$, it is natural to generalize it by considering also other values of this parameter. For instance, the case of complex shadow $\mathcal{P}_A = \mathcal{P}_A^{\mathbb{C}}$ corresponds to the case $k_i = 1$. This fact implies that the real shadow of an extended matrix is equivalent to the complex shadow,

$$\mathcal{P}_{A \otimes \mathbb{I}_2}^{\mathbb{R}}(x) = \mathcal{P}_A(x). \quad (126)$$

This result allows us to consider the real shadow of a real symmetric matrix C of an even size $2N$ as a generalization of the B spline, which is recovered, if each eigenvalue is doubly degenerated. In general, each of N knot points of the standard B -spline can be splitted into two halves, and each eigenvalue λ_i of C can be considered as a 'half of the knot point', as $2N$ points $\{\lambda_i\}_{i=1}^{2N}$ determine the generalized B -spline equal to the real shadow of C .

Analyzing the generalized Dirichlet distribution one needs not to restrict the attention to parameters k_i equal to $1/2$ or 1 . For instance, one can consider the shadow of a matrix of an even order with respect to quaternion states which corresponds to the Dirichlet distribution with all parameters equal, $k_i = 2$, see Appendix A.

As another example of the restricted shadow we analyzed entangled shadow of a matrix of an order N equal to a composite number. As before we distinguish the shadow with respect to complex (or real) maximally entangled states. Note that these probability distributions in general are supported on non-convex sets. In the simplest case of $N = 4$ we found explicit formulae for the complex and real entangled shadows by relating it to the real shadows of suitably transformed matrices. As such shadows visualize projection of the set of complex/real maximally entangled states onto a plane [11] it is likely to expect that such tools will be useful in studying the structure of the set of maximally entangled states.

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Appendix A. Quaternion shadow

In the case of the real shadow one considers random normalized real vectors with distribution invariant to orthogonal transformations. This distribution is induced by a Haar measure on the orthogonal group. In a similar fashion, one can introduce the quaternion shadow, $\mathcal{P}_A^{\mathbb{H}}$, defined as a probability distribution of expectation values taken among random normalized quaternion vectors, with distribution invariant with respect to symplectic operations.

An N -vector with quaternion \mathbb{H} entries is replaced by a $2N \times 2$ complex matrix, where $a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is mapped to

$$\nu(a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) = \begin{bmatrix} a_0 + a_1\mathbf{i} & a_2 + a_3\mathbf{i} \\ -a_2 + a_3\mathbf{i} & a_0 - a_1\mathbf{i} \end{bmatrix}. \quad (\text{A.1})$$

We use ν to also indicate the map N -vectors to $2N \times 2$ matrices. Suppose A is a $2N \times 2N$ complex Hermitian matrix. Consider the numerical range-type map from the unit sphere in \mathbb{H}^N to \mathbb{R} :

$$\xi \mapsto \frac{1}{2} \text{Tr} \left(\nu(\xi)^\dagger A \nu(\xi) \right). \quad (\text{A.2})$$

Note $\nu(\xi)^\dagger A \nu(\xi)$ is a 2×2 complex Hermitian matrix. By direct computation we find that the same values are obtained if A is transformed as follows:

$$q \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} (a_{11} + \overline{a_{22}}) & (a_{12} - \overline{a_{21}}) \\ (a_{21} - \overline{a_{12}}) & (\overline{a_{11}} + a_{22}) \end{bmatrix}, \quad (\text{A.3})$$

and this map is applied to each of the N^2 2×2 blocks of A . Note that these blocks correspond to quaternions of the form

$$\begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix}. \quad (\text{A.4})$$

Then the numerical range and shadow can be interpreted as those of an $N \times N$ quaternionic Hermitian matrix. The probability density is Dirichlet, parameter 2, using N real eigenvalues. The transformed matrix $q(A)$ has (duplicates) pairs of eigenvalues.

Here is a trivial example:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, q(A) = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{bmatrix}. \quad (\text{A.5})$$

The eigenvalues of A are $\pm \frac{1}{2} \pm \frac{1}{2} \sqrt{5}$ and the eigenvalues of $q(A)$ are $-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$.

Consider $M_2(\mathbb{C})$ as an eight-dimensional vector space over \mathbb{R} with the basis:

$$\begin{aligned} \mathbf{1} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{i} = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix}, \mathbf{j} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \mathbf{k} = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}, \\ \zeta &= \begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{bmatrix}, \zeta \mathbf{i} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \zeta \mathbf{j} = \begin{bmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{bmatrix}, \zeta \mathbf{k} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \end{aligned} \quad (\text{A.6})$$

(Pauli matrices). The basis is orthonormal with the inner product

$$\langle \alpha, \beta \rangle = \frac{1}{2} \operatorname{Re} \operatorname{Tr} (\alpha \beta^\dagger), \quad (\text{A.7})$$

and $\langle \alpha, \beta \rangle = \langle \alpha^\dagger, \beta^\dagger \rangle$. Then $q(\alpha) = \langle \alpha, \mathbf{1} \rangle \mathbf{1} + \langle \alpha, \mathbf{i} \rangle \mathbf{i} + \langle \alpha, \mathbf{j} \rangle \mathbf{j} + \langle \alpha, \mathbf{k} \rangle \mathbf{k}$ and $q(\alpha^\dagger) = q(\alpha)^\dagger$. Set $\mathbb{H} = \operatorname{span}_{\mathbb{R}} \{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ (a $*$ -subalgebra). Now suppose $(\alpha_{ij})_{i,j=1}^N$ is a Hermitian matrix with entries in $M_2(\mathbb{C})$ (that is, $\alpha_{ji} = \alpha_{ij}^\dagger$), and $(\beta_i)_{i=1}^N$ is a vector with entries in \mathbb{H} .

Lemma 23. *The following equality holds*

$$\operatorname{Tr} \left(\sum_{i,j=1}^N \beta_i^\dagger \alpha_{ij} \beta_j \right) = \operatorname{Tr} \left(\sum_{i,j=1}^N \beta_i^\dagger q(\alpha_{ij}) \beta_j \right). \quad (\text{A.8})$$

Proof. Break up the sum into $i = j$ and $i < j$ parts. Then $\operatorname{Tr} (\beta_i^\dagger \alpha_{ii} \beta_i) \in \mathbb{R}$ and

$$\begin{aligned} \operatorname{Tr} (\beta_i^\dagger \alpha_{ii} \beta_i) &= \operatorname{Tr} (\alpha_{ii} \beta_i \beta_i^\dagger) = 2 \langle \alpha_{ii}, \beta_i \beta_i^\dagger \rangle \\ &= 2 \langle q(\alpha_{ii}), \beta_i \beta_i^\dagger \rangle = \operatorname{Tr} (\beta_i^\dagger q(\alpha_{ii}) \beta_i). \end{aligned} \quad (\text{A.9})$$

For $i < j$ consider the typical term

$$\begin{aligned} \operatorname{Tr} (\beta_i^\dagger \alpha_{ij} \beta_j) + \operatorname{Tr} (\beta_j^\dagger \alpha_{ji} \beta_i) &= \operatorname{Tr} (\beta_i^\dagger \alpha_{ij} \beta_j) + \operatorname{Tr} (\beta_j^\dagger \alpha_{ji}^\dagger \beta_i) \\ &= 2 \operatorname{Re} \operatorname{Tr} (\beta_i^\dagger \alpha_{ij} \beta_j) = 2 \operatorname{Re} \operatorname{Tr} (\alpha_{ij} \beta_j \beta_i^\dagger) \\ &= 4 \langle \alpha_{ij}, \beta_i \beta_j^\dagger \rangle = 4 \langle q(\alpha_{ij}), \beta_i \beta_j^\dagger \rangle \\ &= \operatorname{Tr} (\beta_i^\dagger q(\alpha_{ij}) \beta_j) + \operatorname{Tr} (\beta_j^\dagger q(\alpha_{ji}) \beta_i), \end{aligned} \quad (\text{A.10})$$

because $q(\alpha_{ji}) = q(\alpha_{ij}^\dagger) = q(\alpha_{ij})^\dagger$. This proves the claim. ■

Every quaternion Hermitian matrix can be diagonalized with symplectic operations, thus when studying quaternion numerical shadow of Hermitian matrices, without loss of generality, we can consider only diagonal matrices with real elements on the diagonal. We note, that for such quaternion matrices, the representation on a block complex matrices gives us $\nu(A) = A \otimes \mathbf{1}_2$. Combining this with relation (126), we may write the following chain of equalities

$$\mathcal{P}_{A \otimes \mathbf{1}_4}^{\mathbb{R}} = \mathcal{P}_{A \otimes \mathbf{1}_2} = \mathcal{P}_A^{\mathbb{H}}. \quad (\text{A.11})$$

Appendix B. Proofs

Proof of Lemma 3. Indeed (set $t_N := 1 - \sum_{i=1}^{N-1} t_i$)

$$\begin{aligned} \mathcal{E} \left[(1 - rX)^{-\tilde{k}} \right] &= \sum_{n=0}^{\infty} \frac{\binom{\tilde{k}}{n}}{n!} \int_{\mathbb{T}_{N-1}} \left(\sum_{i=1}^N a_i t_i \right)^n d\mu_{\mathbf{k}} \\ &= \sum_{n=0}^{\infty} \frac{\binom{\tilde{k}}{n}}{n!} r^n \sum_{\alpha \in \mathbb{N}_0^N, |\alpha|=n} \binom{n}{\alpha} \frac{1}{\binom{\tilde{k}}{|\alpha|}} \prod_{i=1}^N (k_i)_{\alpha_i} \\ &= \sum_{\alpha \in \mathbb{N}_0^N} r^{|\alpha|} \prod_{i=1}^N \frac{(k_i)_{\alpha_i}}{\alpha_i!} = \prod_{i=1}^N (1 - ra_i)^{-k_i}. \end{aligned} \quad (\text{B.1})$$

We used the negative binomial theorem and the multinomial theorem with the multinomial coefficient $\binom{n}{\alpha} = \frac{n!}{\alpha!}$. ■

Proof of Proposition 6. The required value is the integral of $d\mu_{\mathbf{k}}$ over the simplex with vertices $(0, 0, \dots, 0), \xi_i(x)$ for $1 \leq i \leq N-1$. Set $\xi'_i(x) = \frac{a_N - x}{a_N - a_i}$. Change variables to $t_i = \xi'_i(x) s_i$, then

$$\begin{aligned} 1 - F(x) &= \frac{\Gamma(\tilde{k})}{\Gamma(k_N)} \prod_{i=1}^{N-1} \frac{\xi'_i(x)^{k_i}}{\Gamma(k_i)} \int_{\mathbb{T}_{N-1}} \prod_{i=1}^{N-1} s_i^{k_i-1} \left(1 - \sum_{i=1}^{N-1} \xi'_i(x) s_i \right)^{k_N-1} ds_1 \dots ds_{N-1} \\ &= \frac{\Gamma(\tilde{k})}{\Gamma(k_N) \Gamma(\tilde{k} - k_N)} \prod_{i=1}^{N-1} \xi'_i(x)^{k_i} \sum_{\alpha \in \mathbb{N}_0^{N-1}} \frac{(1 - k_N)_{|\alpha|}}{\binom{\tilde{k} - k_N}{|\alpha|+1}} \prod_{i=1}^{N-1} \frac{(k_i)_{\alpha_i}}{\alpha_i!} \xi'_i(x)^{\alpha_i}. \end{aligned} \quad (\text{B.2})$$

The negative binomial series converges when $0 \leq \xi'_i(x) < 1$ for all i , that is, $a_{N-1} < x \leq a_N$. Now replace $\xi'_i(x)$ by $\frac{a_N - x}{a_N - a_i}$ to obtain the stated formula. ■

Proof of Lemma 10. The proof follows the method described in Henrici [14, vol. 1, p. 555]. Use the notation from equation (14). Set $f_i(r) = \prod_{j \neq i} (1 - ra_j)^{-k}$, then

$$\prod_{j=1}^N (1 - ra_j)^{-k} = \sum_{j=1}^k \frac{\beta_{ij}}{(1 - ra_i)^j} + q_i(r) f_i(r), \quad (\text{B.3})$$

where $q_i(r)$ is a polynomial. Multiply the equation by $(1 - ra_i)^k$ to obtain

$$f_i(r) = \sum_{j=1}^k \beta_{ij} (1 - ra_i)^{k-j} + (1 - ra_i)^k q_i(r) f_i(r). \quad (\text{B.4})$$

Apply $\left(\frac{d}{dr}\right)^m$ to both sides and set $r = \frac{1}{a_i}$. This cancels out every term on the right side except for $j = k - m$; this term becomes $\beta_{i,k-m} (-a_i)^m m!$. By the

generalized product rule

$$\left(\frac{d}{dr}\right)^m f_i(r) = \sum_{\alpha \in \mathbb{N}_0^N, |\alpha|=m, \alpha_i=0} \binom{m}{\alpha} \prod_{j=1, j \neq i}^N (k)_{\alpha_j} a_j^{\alpha_j} (1 - r a_j)^{-k-\alpha_j}. \quad (\text{B.5})$$

Set $r = \frac{1}{a_i}$ to get the stated values of $\beta_{i,k-m}$ (note $\left(1 - \frac{a_j}{a_i}\right)^{-k-\alpha_j} = a_i^{k+\alpha_i} (a_i - a_j)^{-k-\alpha_j}$).

■

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